

SIGN-CHANGING TWO-PEAK SOLUTIONS FOR AN ELLIPTIC FREE BOUNDARY PROBLEM RELATED TO CONFINED PLASMAS

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ABSTRACT. By a perturbative argument, we construct solutions for a plasma-type problem with two opposite-signed sharp peaks at levels 1 and $-\gamma$, respectively, where $0 < \gamma < 1$. We establish some physically relevant qualitative properties for such solutions, including the connectedness of the level sets and the asymptotic location of the peaks as $\gamma \rightarrow 0^+$.

1. INTRODUCTION

Motivated by the description of equilibrium states for plasmas in a tokamak [21], we consider the following problem:

$$\begin{cases} -\varepsilon^2 \Delta u = (u-1)_+ - (-u-\gamma)_+ & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where γ is a positive constant, $\varepsilon > 0$ is a small positive number and $\Omega \subset \mathbb{R}^2$ is a smooth bounded domain. Problem (1.1) generalizes the classical plasma problem:

$$\begin{cases} -\varepsilon^2 \Delta u = (u-1)_+ & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

as derived in [21] and extensively analyzed in [6, 7, 17, 21], to the case of a nonlinearity indefinite in sign. From the physical point of view, problem (1.1) corresponds to the case where the tokamak contains two plasmas ionized with charges 1 and $-\gamma$, respectively. The unknown function u corresponds to the magnetic potential and $\varepsilon > 0$ is a constant depending on the constitution of the plasmas. Problem (1.1) admits a variational characterization. Indeed, solutions to (1.1) correspond to critical points for the functional

$$I_\varepsilon(u) = \frac{\varepsilon^2}{2} \int_\Omega |\nabla u|^2 - \frac{1}{2} \int_\Omega [(u-1)_+]^2 - \frac{1}{2} \int_\Omega [(-u-\gamma)_+]^2, \quad u \in H_0^1(\Omega).$$

Our first aim in this article is to construct sign-changing solutions with exactly two sharp peaks, via a perturbative Lyapunov-Schmidt argument as developed in [10, 14]. Our next aim, containing the more innovative aspects, is to derive some new qualitative properties of solutions, including the connectedness of the level sets and the asymptotic location of the peaks as $\gamma \rightarrow 0^+$.

We recall that problems of the form $-\Delta u = f(u)$ in Ω , $u = 0$ on $\partial\Omega$ are also of central interest in the context of steady incompressible Euler flows, see, e.g., [9, 11, 20] and the

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references therein. In this context, such sign-changing solutions are related to the Mallier-Maslowe counter rotating vortices [4, 18, 19]. Sign-changing solutions with exactly two peaks were also constructed for a two-dimensional elliptic problem with large exponent in [15].

Unlike the above mentioned models, problem (1.1) involves the nonlinearity $f(t) = (t - 1)_+ - (-t - \gamma)_+$ which is only Lipschitz continuous. Consequently, major technical difficulties arise due to the “free boundaries” $\{u = 1\}$ and $\{u = -\gamma\}$, particularly in establishing some regularity properties that are essential in order to obtain solutions as critical points of continuously differentiable functionals. Such difficulties were overcome in [10] in the case of one-sided peak solutions. A key ingredient to this end is to establish that such free boundaries have zero measure, see Proposition 5.1 below. Here, we shall obtain this key property by a new simple *ad hoc* argument involving the Faber-Krahn inequality, suitably tailored to the case of two-peak sign-changing solutions. We note that our perturbative construction does not provide any variational characterization of solutions, which is often employed in establishing the zero measure of free boundaries. Our arguments also suggest relations to the “twisted eigenfunctions”, recently analyzed in [12, 13].

After constructing the peak solutions in Theorem 2.1, we analyze the asymptotic location of the peaks as $\gamma \rightarrow 0^+$. Roughly speaking, our result states that for the physically relevant solutions corresponding to minima of the Kirchhoff-Routh Hamiltonian, as $\gamma \rightarrow 0^+$, the positive peak approaches a harmonic center $\underline{z} \in \Omega$ and the negative peak escapes to a point $\bar{p} \in \partial\Omega$ which maximizes the outward normal derivative of the Green’s function with pole at \underline{z} . This type of property was introduced in [19], where the case of a convex domain Ω is considered. Here, we remove the convexity assumption on Ω , see Theorem 2.2.

This article is organized as follows. In Section 2 we introduce some notation and we precisely state the main results. In Section 3 we define the ansatz for solutions in terms of the basic cell functions $U_{\varepsilon,a,z}$ defined in (2.2). We also establish some necessary estimates for the approximate solutions. Section 4 contains the linear theory and the Lyapunov-Schmidt reduction by which we obtain a solution u_ε to a “projected problem” for every fixed choice of peak points $z_1, z_2 \in \Omega$, $z_1 \neq z_2$. Thus, we reduce problem (1.1) to a four-dimensional problem. The results in this section rely on an approach introduced [10, 14]. Therefore, some of the proofs in this section are only outlined for the reader’s convenience. In Section 5 we prove the crucial zero-measure property for the free boundaries $\{u_\varepsilon = 1\}$ and $\{u_\varepsilon = -\gamma\}$, where u_ε is the solution to the projected problem obtained in Section 4. In Section 6 we insert u_ε into the variational functional for (1.1) and we check that critical points for the resulting function $K_\varepsilon(z_1, z_2)$ yield the desired solutions to the full equation (1.1); we also establish the connectedness of the level sets thus establishing Theorem 2.1. In Section 7 we analyze the limit profile of the “minimal” solutions as $\gamma \rightarrow 0^+$, as stated in Theorem 2.2.

2. STATEMENT OF THE MAIN RESULTS

In order to state our results precisely, we introduce some notation. Let $s > 0$ be defined by $s^2 = j_0^{(1)}$, where $j_0^{(1)} \cong 2.405$ is the first zero of J^0 , the first Bessel function of the first kind. Then, the first eigenvalue of $-\Delta$ in $B_s(0)$ subject to zero Dirichlet boundary conditions is $\lambda_1(B_s(0)) = 1$. Let $\varphi_1 > 0$ be the first eigenfunction of $-\Delta$ in $B_s(0)$ satisfying $\varphi_1(0) = 1$. Namely, φ_1 satisfies:

$$\begin{cases} -\Delta\varphi_1 = \varphi_1 & \text{in } B_s(0) \\ \varphi_1 = 0 & \text{on } \partial B_s(0) \\ \varphi_1(0) = 1. \end{cases}$$

For simplicity, in what follows we identify φ with its radial profile; namely, we denote $\varphi(s) = \varphi(x)|_{|x|=s} = J^0(s)$.

We denote by U the “basic cell function” defined on the whole space \mathbb{R}^2 by

$$U(x) = \begin{cases} \varphi_1(x), & \text{if } 0 \leq |x| < s \\ s|\varphi'(s)| \ln \frac{s}{|x|}, & \text{if } |x| \geq s. \end{cases} \quad (2.1)$$

We note that any constant multiple of U satisfies the equation

$$-\Delta u = u_+ \quad \text{on } \mathbb{R}^2.$$

We shall obtain the desired peak solutions as perturbations of an approximate solution defined in terms of suitably rescaled translations of U . More precisely, let $R > 0$ be a sufficiently large constant so that

$$R > \text{diam } \Omega$$

and consequently $\Omega \subset B_R(z)$ for all $z \in \Omega$. We define, for any $\varepsilon, a > 0$:

$$U_{\varepsilon,a}(x) = a + \frac{a}{\ln \frac{R}{s\varepsilon}} U\left(\frac{x}{\varepsilon}\right).$$

Then $U_{\varepsilon,a}$ satisfies the Dirichlet problem

$$\begin{cases} -\varepsilon^2 \Delta u = (u - a)_+ & \text{in } B_R(0) \\ u = 0 & \text{on } \partial B_R(0). \end{cases}$$

For any $z \in \mathbb{R}^2$ we set

$$U_{\varepsilon,a,z}(x) = U_{\varepsilon,a}(x - z). \quad (2.2)$$

Thus, we obtain the family of cell functions $\{U_{\varepsilon,a,z} : \varepsilon, a > 0, z \in \mathbb{R}^2\}$. We note that as $\varepsilon \rightarrow 0$ the function $U_{\varepsilon,a,z}$ develops a sharp peak at the point z at level a . Moreover, as $\varepsilon \rightarrow 0$, we have $U_{\varepsilon,a,z} \rightarrow 0$ in $L_{\text{loc}}^\infty(\Omega \setminus \{z\})$ and $U_{\varepsilon,a,z}(z) \rightarrow a$.

We denote by \bar{G} the Green's function defined by

$$\begin{cases} -\Delta_x \bar{G}(x, z) = 2\pi \delta_z & \text{in } \Omega \\ \bar{G}(x, z) = 0 & \text{on } \partial\Omega, \end{cases} \quad (2.3)$$

and by $g(x, z)$ the regular part of \bar{G} , defined by

$$\begin{cases} -\Delta_x g(x, z) = 0 & \text{in } \Omega \\ g(x, z) = \ln \frac{R}{|x - z|} & \text{on } \partial\Omega. \end{cases}$$

We note that $\bar{G}(x, z) = \ln \frac{R}{|x - z|} - g(x, z)$ and, by the choice of R , we have $g(x, z) \geq 0$ for all $x, z \in \Omega$. We denote by $h : \Omega \rightarrow \mathbb{R}$ the Robin's function:

$$h(z) = g(z, z).$$

We remark that with this notation we have

$$h(z) \geq 0 \text{ for all } z \in \Omega, \quad h(z) \rightarrow +\infty \text{ as } z \rightarrow \partial\Omega. \quad (2.4)$$

We set

$$\mathcal{M} = \{(x, y) \in \Omega \times \Omega : x \neq y\}.$$

We denote by \mathcal{H}_γ the Kirchhoff-Routh type Hamiltonian (see, e.g., [20]) defined by

$$\mathcal{H}_\gamma(z_1, z_2) = h(z_1) + 2\gamma \bar{G}(z_1, z_2) + \gamma^2 h(z_2)$$

for $(z_1, z_2) \in \mathcal{M}$. We note that \mathcal{H}_γ is bounded from below on \mathcal{M} . We denote by $P : H^1(\Omega) \rightarrow H_0^1(\Omega)$ the standard projection operator and we denote by $\text{cat } \mathcal{M}$ the Lusternik-Schnirelmann category of \mathcal{M} .

Our aim is to establish the following results.

Theorem 2.1. *For every $\gamma > 0$ there exists $\varepsilon_\gamma > 0$ such that for all $0 < \varepsilon < \varepsilon_\gamma$ problem (1.1) admits at least $\text{cat} \mathcal{M}$ sign-changing two-peak solutions of the form*

$$u_\varepsilon^i = PU_{\varepsilon, a_{\varepsilon,1}^i, z_{\varepsilon,1}^i} - PU_{\varepsilon, a_{\varepsilon,2}^i, z_{\varepsilon,2}^i} + \omega_\varepsilon^i, \quad (2.5)$$

$i = 1, \dots, \text{cat} \mathcal{M}$, where $\|\omega_\varepsilon^i\|_{L^\infty(\Omega)} = O(\varepsilon/|\ln \varepsilon|)$, $a_{\varepsilon,1}^i = 1 + O(|\ln \varepsilon|^{-1})$, $a_{\varepsilon,2}^i = \gamma + O(|\ln \varepsilon|^{-1})$ and $(z_{\varepsilon,1}^i, z_{\varepsilon,2}^i) \in \mathcal{M}$, $(z_{\varepsilon,1}^i, z_{\varepsilon,2}^i) \rightarrow (z_1^i, z_2^i) \in \mathcal{M}$, with (z_1^i, z_2^i) a critical point for \mathcal{H}_γ .

Moreover, the following properties hold.

- (i) *The level sets $\{u_\varepsilon^i > 1\}$ and $\{u_\varepsilon^i < -\gamma\}$, $i = 1, \dots, \text{cat} \mathcal{M}$, are connected.*
- (ii) *If $\gamma = 1$, then problem (1.1) admits at least $\text{cat}[\mathcal{M}/((x, y) \sim (y, x))]$ pairs of sign-changing solutions of the form (2.5), $i = 1, \dots, \text{cat}[\mathcal{M}/((x, y) \sim (y, x))]$.*

It may be worth observing that, although $L^\infty(\Omega)$ -bounded, the family of solutions $\{u_\varepsilon\}_{\varepsilon>0}$ obtained in Theorem 2.1 presents a lack of compactness in C^α -sense.

In order to state our second result, we recall that $\underline{z} \in \Omega$ is a harmonic center of Ω if it is a minimum point of the Robin's function $h(z) = g(z, z)$, see [3]. If Ω is convex, then \underline{z} is unique [8].

Theorem 2.2. *For every $\gamma > 0$, let u_ε^γ be a family of solutions to (1.1), as obtained in Theorem 2.1, with peak points at $(z_{1,\varepsilon}^\gamma, z_{2,\varepsilon}^\gamma) \rightarrow (z_1^\gamma, z_2^\gamma)$ as $\varepsilon \rightarrow 0^+$, where $\mathcal{H}_\gamma(z_1^\gamma, z_2^\gamma) = \min_{\mathcal{M}} \mathcal{H}_\gamma$. For every $\eta > 0$ there exist $\gamma_\eta > 0$ and $\varepsilon(\gamma_\eta) > 0$ such that $d(z_{1,\varepsilon}^\gamma, \underline{z}) < \eta$, $d(z_{2,\varepsilon}^\gamma, p_0) < \eta$ for all $\gamma \in (0, \gamma_\eta)$, $\varepsilon \in (0, \varepsilon(\gamma_\eta))$, where \underline{z} is a harmonic center for Ω and $\bar{p} \in \partial\Omega$ satisfies $\partial_\nu G(\underline{z}, \bar{p}) = \max_{p \in \partial\Omega} \partial_\nu G(\underline{z}, p)$.*

Theorem 2.2 implies that for peak solutions corresponding to minima of \mathcal{H}_γ , for small values of γ the positive peak is approximately located at a harmonic center $\underline{z} \in \Omega$ and the negative peak is near the boundary $\partial\Omega$.

Notation. Henceforth, for any measurable set $S \subset \mathbb{R}^2$ we denote by mS the two-dimensional Lebesgue measure of S and by 1_S the characteristic function of S . All integrals are taken with respect to the Lebesgue measure; when the integration variable is clear from the context, we may omit it. We denote $\|\cdot\|_p = \|\cdot\|_{L^p(\Omega)}$. We denote by $C > 0$ a general constant, whose actual value may vary from line to line. Let $\Omega \subset \mathbb{R}^2$ be an open set, for a given function V defined on $\Omega \times \Omega$ and $Z = (z_1, z_2) \in \Omega \times \Omega$ we use the notation $\frac{\partial V}{\partial z_{i,j}}$ to denote the partial derivative of V with respect to the component j of the variable i .

3. ANSATZ AND PROPERTIES OF APPROXIMATE SOLUTIONS

The aim of this section is to define suitable approximate solutions W_ε to problem (1.1) in terms of the basic cell functions $U_{\varepsilon,a,z}$, for $\varepsilon, a > 0$, $z \in \mathbb{R}^2$, defined in (2.2). We also establish some basic properties of W_ε which will be needed in the sequel.

We recall from [10] that for any $a > 0$ and $\varepsilon > 0$, the problem

$$\begin{cases} -\varepsilon^2 \Delta u = (u - a)_+ & \text{in } B_R(0) \\ u = 0 & \text{on } \partial B_R(0) \end{cases} \quad (3.1)$$

has as unique solution $U_{\varepsilon,a}(x)$ defined by

$$U_{\varepsilon,a}(x) = a + \frac{a}{\ln \frac{R}{s\varepsilon}} U\left(\frac{x}{\varepsilon}\right) = \begin{cases} a \left[1 + k_\varepsilon \varphi_1\left(\frac{x}{\varepsilon}\right)\right] & \text{if } |x| \leq s\varepsilon \\ \frac{a}{\ln \frac{R}{s\varepsilon}} \ln \frac{R}{|x|} & \text{if } s\varepsilon \leq |x| \leq R, \end{cases} \quad (3.2)$$

where

$$k_\varepsilon := \frac{1}{s\varphi_1'(s) \ln \frac{s\varepsilon}{R}} > 0.$$

We note that $U_{\varepsilon,a} > 0$ in $B_R(0)$ and $k_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$. Moreover, we have that $U_{\varepsilon,a}(x) \rightarrow 0$ in $L_{\text{loc}}^\infty(B_R(0) \setminus \{0\})$. Correspondingly, we obtain the family $\{U_{\varepsilon,a,z} : z \in \Omega, \varepsilon, a > 0\}$ of peak solutions defined in (2.2). Since the functions $U_{\varepsilon,a,z}(x)$ do not satisfy the zero Dirichlet boundary condition on $\partial\Omega$, as usual we define their projections on $H_0^1(\Omega)$. We recall that, given $u \in H^1(\Omega)$, the projection of u into $H_0^1(\Omega)$, denoted Pu , is the unique weak solution to

$$\Delta Pu = \Delta u \quad \text{in } \Omega, \quad Pu = 0 \quad \text{on } \partial\Omega.$$

In particular, in view of (3.1), $PU_{\varepsilon,a,z}(x)$ satisfies

$$\begin{cases} -\varepsilon^2 \Delta PU_{\varepsilon,a,z} = (U_{\varepsilon,a,z} - a)_+ & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover, for all $\varepsilon > 0$ sufficiently small so that $B_{s\varepsilon}(z) \subset \Omega$, we have that $U_{\varepsilon,a,z}$ coincides with a Newtonian potential on $\partial\Omega$, namely

$$U_{\varepsilon,a,z}(x) = \frac{a}{\ln \frac{R}{s\varepsilon}} \ln \frac{R}{|x - z|} \quad \forall x \in \partial\Omega.$$

The following is readily checked.

Lemma 3.1. *Let $\varepsilon > 0$ be such that $B_{s\varepsilon}(z) \subset \Omega$. We have, for every $z \in \Omega$:*

$$PU_{\varepsilon,a,z}(x) = U_{\varepsilon,a,z}(x) - \frac{a}{\ln \frac{R}{s\varepsilon}} g(x, z), \quad \forall x \in \Omega \quad (3.3)$$

and more precisely

$$PU_{\varepsilon,a,z}(x) = \begin{cases} a \left[1 + k_\varepsilon \varphi_1 \left(\frac{x - z}{\varepsilon} \right) - \frac{g(x, z)}{\ln \frac{R}{s\varepsilon}} \right], & \text{if } x \in B_{s\varepsilon}(z) \\ \frac{a}{\ln \frac{R}{s\varepsilon}} \bar{G}(x, z), & \text{if } x \in \Omega \setminus B_{s\varepsilon}(z). \end{cases} \quad (3.4)$$

We seek solutions whose form is approximately the difference of two cell functions of the form (2.2). Let $Z = (z_1, z_2) \in \mathcal{M}$. We make the following

Ansatz. The solution u to problem (1.1) is of the form:

$$\begin{aligned} u &:= W_\varepsilon + \omega_\varepsilon \\ W_\varepsilon &:= PU_{\varepsilon,z_1,a_1} - PU_{\varepsilon,z_2,a_2}. \end{aligned} \quad (3.5)$$

Henceforth, we denote $U_i = U_{\varepsilon,z_i,a_i}$, $i = 1, 2$. We assume that

$$d(z_i, \partial\Omega) \geq \delta > 0, \quad i = 1, 2; \quad |z_1 - z_2| \geq \delta \quad (3.6)$$

for some $\delta > 0$.

Choice of $a_i = a_{\varepsilon,i}$, $i = 1, 2$. The pair of constants $(a_{\varepsilon,1}, a_{\varepsilon,2})$ is chosen as the (unique) solution to the following linear system:

$$\begin{cases} \left(1 - \frac{g(z_1, z_1)}{\ln \frac{R}{s\varepsilon}} \right) a_1 - \frac{\bar{G}(z_1, z_2)}{\ln \frac{R}{s\varepsilon}} a_2 = 1 \\ -\frac{\bar{G}(z_1, z_2)}{\ln \frac{R}{s\varepsilon}} a_1 + \left(1 - \frac{g(z_2, z_2)}{\ln \frac{R}{s\varepsilon}} \right) a_2 = \gamma. \end{cases} \quad (3.7)$$

Namely,

$$a_{\varepsilon,1} = \frac{1 - \frac{g(z_2, z_2)}{\ln \frac{R}{s\varepsilon}} - \bar{G}(z_1, z_2)\gamma}{\left(1 - \frac{g(z_1, z_1)}{\ln \frac{R}{s\varepsilon}}\right)\left(1 - \frac{g(z_2, z_2)}{\ln \frac{R}{s\varepsilon}}\right) - \frac{\bar{G}(z_1, z_2)^2}{\left(\ln \frac{R}{s\varepsilon}\right)^2}} = 1 + O\left(\frac{1}{|\ln \varepsilon|}\right)$$

$$a_{\varepsilon,2} = \frac{\left(1 - \frac{g(z_1, z_1)}{\ln \frac{R}{s\varepsilon}}\right)\gamma - \bar{G}(z_1, z_2)\gamma}{\left(1 - \frac{g(z_1, z_1)}{\ln \frac{R}{s\varepsilon}}\right)\left(1 - \frac{g(z_2, z_2)}{\ln \frac{R}{s\varepsilon}}\right) - \frac{\bar{G}(z_1, z_2)^2}{\left(\ln \frac{R}{s\varepsilon}\right)^2}} = \gamma + O\left(\frac{1}{|\ln \varepsilon|}\right).$$

In order to justify the choice of $a_i = a_{\varepsilon,i}$, $i = 1, 2$ in terms of $\gamma, \varepsilon, z_1, z_2$ and Ω , we consider the “error term”

$$\ell_\varepsilon(x) := \varepsilon^2 \Delta W_\varepsilon + (W_\varepsilon - 1)_+ - (-W_\varepsilon - \gamma)_+$$

and we estimate ℓ_ε near z_1 and near z_2 . Let us first observe that in order to approximate a solution of (1.1) it is reasonable to choose $a_1 \simeq 1$ and $a_2 \simeq \gamma$ therefore we suppose $0 < a_i < K$ for a given $K > 0$.

In $B_{s\varepsilon}(z_1)$, in view of Lemma 3.1, we have

$$\begin{aligned} \ell_\varepsilon(x) = & -(U_1 - a_1)_+ + \left(U_1 - \frac{a_1}{\ln \frac{R}{s\varepsilon}} g(x, z_1) - \frac{a_2}{\ln \frac{R}{s\varepsilon}} \bar{G}(x, z_2) - 1\right)_+ \\ & + (U_2 - a_2)_+ - \left(\frac{a_2}{\ln \frac{R}{s\varepsilon}} \bar{G}(x, z_2) - PU_1 - \gamma\right)_+. \end{aligned}$$

We note that $(U_2 - a_2)_+ \equiv 0$ in $B_{s\varepsilon}(z_1)$. Moreover, for sufficiently small $\varepsilon > 0$ we also have $\frac{a_2}{\ln \frac{R}{s\varepsilon}} \bar{G}(x, z_2) - \gamma \leq 0$ in $B_{s\varepsilon}(z_1)$. Since $PU_1 \geq 0$ in Ω , we conclude that

$$\left(\frac{a_2}{\ln \frac{R}{s\varepsilon}} \bar{G}(x, z_2) - PU_1 - \gamma\right)_+ \equiv 0$$

in $B_{s\varepsilon}(z_1)$. It follows that in $B_{s\varepsilon}(z_1)$ we have

$$\ell_\varepsilon(x) = -(U_1 - a_1)_+ + \left(U_1 - \frac{a_1}{\ln \frac{R}{s\varepsilon}} g(x, z_1) - \frac{a_2}{\ln \frac{R}{s\varepsilon}} \bar{G}(x, z_2) - 1\right)_+.$$

Taking $x = z_1$ we fit a_1 by requiring that $\ell_\varepsilon(z_1) = 0$. We consequently derive

$$a_1 = \frac{a_1}{\ln \frac{R}{s\varepsilon}} g(z_1, z_1) + \frac{a_2}{\ln \frac{R}{s\varepsilon}} \bar{G}(z_1, z_2) + 1,$$

which yields the first equation in (3.7). The second equation is obtained similarly.

We note that the Ansatz and the choice of $(a_{\varepsilon,1}, a_{\varepsilon,2})$ show that the interaction between PU_1 and PU_2 is essentially negligible. Moreover, we have the following expansions

$$\begin{aligned} \frac{\partial a_{\varepsilon,i}}{\partial z_{i,h}} &= O\left(\frac{1}{|\log \varepsilon|}\right), \tag{3.8} \\ \frac{\partial U_{\varepsilon, z_i, a_{\varepsilon,i}}(x)}{\partial z_{i,h}} &= \frac{a_{\varepsilon,i}}{\ln \frac{R}{s\varepsilon}} \frac{\partial}{\partial z_{i,h}} U\left(\frac{x - z_i}{\varepsilon}\right) + \frac{U\left(\frac{x - z_i}{\varepsilon}\right)}{\ln \frac{R}{s\varepsilon}} \frac{\partial a_{\varepsilon,i}}{\partial z_{i,h}} \\ &= \begin{cases} \frac{k_\varepsilon a_{\varepsilon,i}}{\varepsilon} \varphi'_1\left(\frac{|x - z_i|}{\varepsilon}\right) \frac{z_{i,h} - x_h}{|x - z_i|} + O\left(\frac{\partial a_{\varepsilon,i}}{\partial z_{i,h}}\right), & \text{if } x \in B_{s\varepsilon}(z_i) \\ \frac{k_\varepsilon a_{\varepsilon,i}}{|x - z_i|} \frac{z_{i,h} - x_h}{|x - z_i|} + O\left(\frac{\partial a_{\varepsilon,i}}{\partial z_{i,h}} \frac{\log \frac{R}{s\varepsilon}}{\log \frac{R}{s\varepsilon}}\right), & \text{if } x \in \Omega \setminus B_{s\varepsilon}(z_i) \end{cases}, \tag{3.9} \end{aligned}$$

where U is defined in (2.1). In view of (3.7) we have the following.

Lemma 3.2. *The following identities hold:*

$$W_\varepsilon(x) \equiv \begin{cases} 1 + a_1 k_\varepsilon \varphi_1\left(\frac{x-z_1}{\varepsilon}\right) - \frac{a_1}{\ln \frac{R}{s\varepsilon}} [g(x, z_1) - g(z_1, z_1)] - \frac{a_2}{\ln \frac{R}{s\varepsilon}} [\bar{G}(x, z_2) - \bar{G}(z_1, z_2)], & \text{in } B_{s\varepsilon}(z_1); \\ -\gamma - a_2 k_\varepsilon \varphi_1\left(\frac{x-z_2}{\varepsilon}\right) + \frac{a_1}{\ln \frac{R}{s\varepsilon}} [\bar{G}(x, z_1) - \bar{G}(z_2, z_1)] + \frac{a_2}{\ln \frac{R}{s\varepsilon}} [g(x, z_2) - g(z_2, z_2)], & \text{in } B_{s\varepsilon}(z_2); \\ \frac{a_1-a_2}{\ln \frac{R}{s\varepsilon}} \bar{G}(z_1, z_2) + \frac{a_1}{\ln \frac{R}{s\varepsilon}} [\bar{G}(x, z_1) - \bar{G}(z_1, z_2)] - \frac{a_2}{\ln \frac{R}{s\varepsilon}} [\bar{G}(x, z_2) - \bar{G}(z_1, z_2)], & \text{otherwise.} \end{cases}$$

Proof. We exploit the explicit expressions of PU_1, PU_2 and the definitions of a_1, a_2 as in (3.7).

In $B_{s\varepsilon}(z_1)$ we have:

$$\begin{aligned} W_\varepsilon(x) &= PU_1(x) - PU_2(x) = a_1 \left[1 + k_\varepsilon \varphi_1\left(\frac{x-z_1}{\varepsilon}\right) - \frac{g(x, z_1)}{\ln \frac{R}{s\varepsilon}} \right] - \frac{a_2}{\ln \frac{R}{s\varepsilon}} \bar{G}(x, z_2) \\ &= a_1 k_\varepsilon \varphi_1\left(\frac{x-z_1}{\varepsilon}\right) + a_1 \left[1 - \frac{g(x, z_1)}{\ln \frac{R}{s\varepsilon}} \right] - \frac{a_2}{\ln \frac{R}{s\varepsilon}} \bar{G}(x, z_2) \\ &= a_1 k_\varepsilon \varphi_1\left(\frac{x-z_1}{\varepsilon}\right) + a_1 \left[1 - \frac{g(z_1, z_1)}{\ln \frac{R}{s\varepsilon}} \right] - \frac{a_2}{\ln \frac{R}{s\varepsilon}} \bar{G}(z_1, z_2) \\ &\quad - \frac{a_1}{\ln \frac{R}{s\varepsilon}} [g(x, z_1) - g(z_1, z_1)] - \frac{a_2}{\ln \frac{R}{s\varepsilon}} [\bar{G}(x, z_2) - \bar{G}(z_1, z_2)] \\ &= 1 + a_1 k_\varepsilon \varphi_1\left(\frac{x-z_1}{\varepsilon}\right) - \frac{a_1}{\ln \frac{R}{s\varepsilon}} [g(x, z_1) - g(z_1, z_1)] - \frac{a_2}{\ln \frac{R}{s\varepsilon}} [\bar{G}(x, z_2) - \bar{G}(z_1, z_2)], \end{aligned}$$

where we used the first equation in (3.7) in the last inequality.

Similarly, in $B_{s\varepsilon}(z_2)$ we have:

$$\begin{aligned} W_\varepsilon(x) &= \frac{a_1}{\ln \frac{R}{s\varepsilon}} \bar{G}(x, z_1) - a_2 \left[1 + k_\varepsilon \varphi_1\left(\frac{x-z_2}{\varepsilon}\right) - \frac{g(x, z_2)}{\ln \frac{R}{s\varepsilon}} \right] \\ &= -a_2 k_\varepsilon \varphi_1\left(\frac{x-z_2}{\varepsilon}\right) - a_2 \left[1 - \frac{g(x, z_2)}{\ln \frac{R}{s\varepsilon}} \right] + \frac{a_1}{\ln \frac{R}{s\varepsilon}} \bar{G}(x, z_1) \\ &= -a_2 k_\varepsilon \varphi_1\left(\frac{x-z_2}{\varepsilon}\right) - a_2 \left[1 - \frac{g(z_2, z_2)}{\ln \frac{R}{s\varepsilon}} \right] + \frac{a_1}{\ln \frac{R}{s\varepsilon}} \bar{G}(z_2, z_1) \\ &\quad + \frac{a_2}{\ln \frac{R}{s\varepsilon}} [g(x, z_2) - g(z_2, z_2)] + \frac{a_1}{\ln \frac{R}{s\varepsilon}} [\bar{G}(x, z_1) - \bar{G}(z_2, z_1)] \\ &= -\gamma - a_2 k_\varepsilon \varphi_1\left(\frac{x-z_2}{\varepsilon}\right) + \frac{a_2}{\ln \frac{R}{s\varepsilon}} [g(x, z_2) - g(z_2, z_2)] + \frac{a_1}{\ln \frac{R}{s\varepsilon}} [\bar{G}(x, z_1) - \bar{G}(z_2, z_1)], \end{aligned}$$

where we used the second equation in (3.7) in the last inequality.

The third identity readily follows observing that in $\Omega \setminus (B_{s\varepsilon}(z_1) \cup B_{s\varepsilon}(z_2))$ we have

$$W_\varepsilon(x) = \frac{a_1}{\ln \frac{R}{s\varepsilon}} \bar{G}(x, z_1) - \frac{a_2}{\ln \frac{R}{s\varepsilon}} \bar{G}(x, z_2).$$

□

The choice of a_1, a_2 in (3.7) also implies the following estimates.

Lemma 3.3. *For any $L > 0$ fixed constant we have, for ε sufficiently small, the following expansions for W_ε :*

$$\begin{aligned} W_\varepsilon(y) - 1 &= U_{\varepsilon, z_1, a_1} - a_1 - \frac{a_1}{\log \frac{R}{s\varepsilon}} \langle Dg(z_1, z_1), y - z_1 \rangle + \frac{a_2}{\log \frac{R}{s\varepsilon}} \langle D\bar{G}(z_1, z_2), y - z_1 \rangle \\ &\quad + O\left(\frac{\varepsilon^2}{|\log \varepsilon|}\right) \end{aligned} \tag{3.10}$$

for $y \in B_{L\varepsilon}(z_1)$;

$$\begin{aligned} -W_\varepsilon(y) - \gamma &= U_{\varepsilon, z_2, a_2} - a_2 - \frac{a_2}{\log \frac{R}{s\varepsilon}} \langle Dg(z_2, z_2), y - z_2 \rangle - \frac{a_1}{\log \frac{R}{s\varepsilon}} \langle D\bar{G}(z_2, z_i), y - z_2 \rangle \\ &\quad + O\left(\frac{\varepsilon^2}{|\log \varepsilon|}\right) \end{aligned} \quad (3.11)$$

for $y \in B_{L\varepsilon}(z_2)$.

Proof. The proof is readily derived from Lemma 3.2 and a Taylor expansion at z_1 and z_2 . \square

Using Lemma 3.3 we now provide a quantitative estimate of the sets where the approximate solution W_ε takes values less than $-\gamma$, between $-\gamma$ and 1 and greater than 1. These estimates will be useful in the study of the linearized problem for the finite dimensional reduction scheme.

Lemma 3.4 (Level set estimates). *There exist $T \gg 1$ and $0 < \sigma < 1$ independent on ε and $0 < \varepsilon_0 \ll 1$ such that for all $0 < \varepsilon < \varepsilon_0$ we have*

$$\begin{aligned} W_\varepsilon(y) - 1 &> 0, \quad y \in B_{s\varepsilon(1-T\varepsilon)}(z_1), \\ -W_\varepsilon(y) - \gamma &> 0, \quad y \in B_{s\varepsilon(1-T\varepsilon)}(z_2) \\ -\gamma < W_\varepsilon(y) < 1, \quad y \in \Omega \setminus (B_{s\varepsilon(1+\varepsilon^\sigma)}(z_1) \cup B_{s\varepsilon(1+\varepsilon^\sigma)}(z_2)). \end{aligned}$$

Proof. We follow the proof of Lemma A.1 in [10]. We start by taking $y \in B_{s\varepsilon(1-T\varepsilon)}(z_1)$. Using (2.3), the monotonicity of φ_1 , a Taylor expansion of φ_1 around s and $\varphi'_1(s) < 0$, we deduce that

$$\begin{aligned} U_{\varepsilon, z_1, a_1}(y) - a_1 &= \frac{1}{\log \frac{R}{s\varepsilon}} \left(\frac{-a_1}{s\varphi'_1(s)} \varphi_1 \left(\frac{|y - z_1|}{\varepsilon} \right) \right) \geq \frac{1}{\log \frac{R}{s\varepsilon}} \left(\frac{-a_1}{s\varphi'_1(s)} \varphi_1(s - sT\varepsilon) \right) \\ &= \frac{1}{\log \frac{R}{s\varepsilon}} \left(\frac{a_1}{s\varphi'_1(s)} [\varphi'_1(s)sT\varepsilon + O(\varepsilon^2)] \right) = \frac{a_1 T\varepsilon}{\log \frac{R}{s\varepsilon}} + O\left(\frac{\varepsilon^2}{|\log \varepsilon|}\right). \end{aligned}$$

Choose $\varepsilon \ll 1$ such that $\gamma_1 = \min\{1/2, \gamma/2\} \leq a_i \leq \max\{3/2, \gamma + 1/2\} = \gamma_2$ and let $M = M(\delta, \gamma) > 0$ such that for any (z_1, z_2) satisfying (3.6), we have

$$M \geq a_1 |Dg(z_1, z_2)| + a_2 |D\bar{G}(z_1, z_2)|.$$

Using (3.10) and the previous estimates we deduce that

$$\begin{aligned} W_\varepsilon(y) - 1 &= U_{\varepsilon, z_1, a_1}(y) - a_1 + \frac{1}{\log \frac{R}{s\varepsilon}} a_2 \langle D\bar{G}(z_1, z_2), y - z_1 \rangle \\ &\quad - \frac{1}{\log \frac{R}{s\varepsilon}} a_1 \langle Dg(z_1, z_1), y - z_1 \rangle + O\left(\frac{\varepsilon^2}{|\log \varepsilon|}\right) \\ &\geq \frac{1}{\log \frac{R}{s\varepsilon}} (a_1 T\varepsilon - Ms\varepsilon(1 - T\varepsilon)) + O\left(\frac{\varepsilon^2}{|\log \varepsilon|}\right) \\ &\geq \frac{\varepsilon}{\log \frac{R}{s\varepsilon}} (\gamma_1 T - Ms) + O\left(\frac{\varepsilon^2}{|\log \varepsilon|}\right) \end{aligned}$$

and the claim follows by choosing $T > \frac{Ms}{\gamma_1}$ and ε_0 sufficiently small.

If $y \in B_{s\varepsilon(1-T\varepsilon)}(z_2)$ we proceed in a similar way using (3.11) and the fact that

$$U_{\varepsilon, z_2, a_2}(y) - a_2 \geq \frac{a_2 T\varepsilon}{\log \frac{R}{s\varepsilon}} + O\left(\frac{\varepsilon^2}{|\log \varepsilon|}\right).$$

To get the estimate of W_ε away from the points z_i we begin by noticing that, if $y \in \Omega \setminus (B_{\varepsilon\bar{\sigma}}(z_1) \cup B_{\varepsilon\bar{\sigma}}(z_2))$, for some $\bar{\sigma} < 1$ to be chosen later, by Lemma 3.1, we can write, for any ε sufficiently small such that $\varepsilon^{\bar{\sigma}} > s\varepsilon$,

$$PU_{\varepsilon, z_i, a_i}(y) = \frac{a_i}{\log \frac{R}{s\varepsilon}} \bar{G}(y, z_i) = \frac{a_i}{\log \frac{s\varepsilon}{R}} \log \frac{|y - z_i|}{R} + O\left(\frac{1}{|\log \varepsilon|}\right).$$

It follows that, by choosing for instance $\bar{\sigma} < \gamma_1/2\gamma_2$ and ε sufficiently small, we can write

$$\begin{aligned} W_\varepsilon(y) - 1 &= \frac{a_1}{\log \frac{s\varepsilon}{R}} \log \frac{|y - z_1|}{R} - \frac{a_2}{\log \frac{s\varepsilon}{R}} \log \frac{|y - z_2|}{R} - 1 + O\left(\frac{1}{|\log \varepsilon|}\right) \\ &\leq 2\gamma_2 \frac{\log \frac{\varepsilon^{\bar{\sigma}}}{R}}{\log \frac{s\varepsilon}{R}} - 1 + O\left(\frac{1}{|\log \varepsilon|}\right) = \gamma_2 \bar{\sigma} \frac{\log \varepsilon}{\log \frac{s\varepsilon}{R}} - 1 + O\left(\frac{1}{|\log \varepsilon|}\right) < 0 \end{aligned}$$

and analogously

$$-W_\varepsilon(y) - \gamma \leq 2\gamma_2 \frac{\log \frac{\varepsilon^{\bar{\sigma}}}{R}}{\log \frac{s\varepsilon}{R}} - \gamma + O\left(\frac{1}{|\log \varepsilon|}\right) = \gamma_2 \bar{\sigma} \frac{\log \varepsilon}{\log \frac{s\varepsilon}{R}} - \gamma + O\left(\frac{1}{|\log \varepsilon|}\right) < 0.$$

Moreover, for y in the annulus $B_{\varepsilon\bar{\sigma}}(z_1) \setminus B_{s\varepsilon(1+T\varepsilon\bar{\sigma})}(z_1)$, using (3.3) for $PU_{\varepsilon, z_1, a_1}(y)$, (3.4) for $PU_{\varepsilon, z_2, a_2}(y)$ and (3.7), we can write, for sufficiently small ε ,

$$\begin{aligned} W_\varepsilon(y) - 1 &= U_{\varepsilon, z_1, a_1}(y) - \frac{a_1}{\log \frac{R}{s\varepsilon}} g(y, z_1) - \frac{a_2}{\log \frac{R}{s\varepsilon}} \bar{G}(y, z_2) - 1 \\ &= U_{\varepsilon, z_1, a_1}(y) - a_1 - \frac{a_1}{\log \frac{R}{s\varepsilon}} (g(y, z_1) - g(z_1, z_1)) - \frac{a_2}{\log \frac{R}{s\varepsilon}} (\bar{G}(y, z_2) - \bar{G}(z_1, z_2)) \\ &\leq \frac{a_1}{\log \frac{s\varepsilon}{R}} \log \frac{|y - z_1|}{R} - a_1 + 2M\gamma_2 \frac{\varepsilon^{\bar{\sigma}}}{\log \frac{R}{s\varepsilon}} \\ &= 2M\gamma_2 \frac{\varepsilon^{\bar{\sigma}}}{\log \frac{R}{s\varepsilon}} - \frac{a_1}{\log \frac{R}{s\varepsilon}} \log \frac{|y - z_1|}{s\varepsilon} \leq 2M\gamma_2 \frac{\varepsilon^{\bar{\sigma}}}{\log \frac{R}{s\varepsilon}} - a_1 \frac{\log(1 + T\varepsilon\bar{\sigma})}{\log \frac{R}{s\varepsilon}} \\ &\leq \frac{\varepsilon^{\bar{\sigma}}}{\log \frac{R}{s\varepsilon}} \left(2M\gamma_2 - \gamma_1 \frac{T}{2}\right) \end{aligned}$$

We can choose $T > \frac{4M\gamma_2}{\gamma_1}$ in order to assure that the last term in the previous inequality is negative. In a similar way we can choose T sufficiently large to have, for any $y \in B_{\varepsilon\bar{\sigma}}(z_2) \setminus B_{s\varepsilon(1+T\varepsilon\bar{\sigma})}(z_2)$,

$$-W_\varepsilon(y) - \gamma \leq 2M\gamma_2 \frac{\varepsilon^{\bar{\sigma}}}{\log \frac{R}{s\varepsilon}} - \frac{a_2}{\log \frac{R}{s\varepsilon}} \log \frac{|y - z_2|}{s\varepsilon} < 0.$$

Finally, to get the claim, we choose $\sigma < \bar{\sigma}$ and ε_0 small enough to satisfy the previous estimates and $B_{s\varepsilon(1+T\varepsilon\bar{\sigma})}(z_i) \subset B_{s\varepsilon(1+\varepsilon^\sigma)}(z_i)$. □

4. REDUCTION TO A FOUR-DIMENSIONAL PROBLEM

Our aim in this section is to reduce problem (1.1) to a four-dimensional problem depending on $Z = (z_1, z_2)$, via a Lyapunov-Schmidt argument; that is, we shall solve a “projection” of problem (1.1) for any given $Z = (z_1, z_2) \in \mathcal{M}$ satisfying (3.6), see Proposition 4.1 below. The content of this section follows [10, 14] closely; therefore, some proofs are only outlined.

Henceforth, we set

$$V_{\varepsilon, Z, j} := PU_{\varepsilon, z_j, a_j},$$

where $a_j = a_j(\varepsilon, Z)$, $j = 1, 2$ are defined in (3.7). With this notation, ansatz (3.5) takes the form

$$u = V_1 - V_2 + \omega_\varepsilon$$

with the convention that $V_j := V_{\varepsilon, Z, j}$, $j = 1, 2$.

We first introduce some function spaces. For $p > 1$ let

$$\begin{aligned} F_{\varepsilon, Z} &= \left\{ v : v \in L^p(\Omega), \int_{\Omega} \frac{\partial V_j}{\partial z_{j,h}} v = 0, j, h = 1, 2 \right\} \\ E_{\varepsilon, Z} &= \left\{ u : u \in W^{2,p}(\Omega) \cap H_0^1(\Omega), \int_{\Omega} \Delta \left(\frac{\partial V_j}{\partial z_{j,h}} \right) u = 0, j, h = 1, 2 \right\}. \end{aligned} \quad (4.1)$$

We define a “localized projection operator” $Q_{\varepsilon} : L^p(\Omega) \rightarrow F_{\varepsilon, Z}$ whose action is supported in $B_{2s\varepsilon}(z_1) \cup B_{2s\varepsilon}(z_2)$. To this end, let $\xi(t) : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth cut-off function satisfying $\xi(t) \equiv 1$ for $0 \leq t \leq 1$, $\xi(t) \equiv 0$ for $t \geq 2$, $0 \leq \xi(t) \leq 1$ and define for $i = 1, 2$

$$\xi_{\varepsilon}^i(y) := \xi \left(\frac{|y - z_i|}{s\varepsilon} \right).$$

For every $u \in L^p(\Omega)$ we set

$$Q_{\varepsilon} u(y) = u(y) - \xi_{\varepsilon}^1(y) \langle \mathbf{b}_1, \nabla_{z_1} U_1(y) \rangle - \xi_{\varepsilon}^2(y) \langle \mathbf{b}_2, \nabla_{z_2} U_2(y) \rangle \quad (4.2)$$

where the vectors $\mathbf{b}_i = (b_{i,1}, b_{i,2}) \in \mathbb{R}^2$ are defined as solution of the linear system

$$\mathbf{A} \cdot (\mathbf{b}_1, \mathbf{b}_2) = \int_{\Omega} u \cdot (\nabla_{z_1} V_1, \nabla_{z_2} V_2)$$

where the 4×4 matrix \mathbf{A} is given by

$$\mathbf{A} := \int_{\Omega} (\nabla_{z_1} V_1, \nabla_{z_2} V_2) \otimes (\xi_{\varepsilon}^1 \nabla_{z_1} U_1, \xi_{\varepsilon}^2 \nabla_{z_2} U_2) dy$$

Equivalently, in coordinates (4.2) takes the form

$$Q_{\varepsilon} u(y) = u(y) - \sum_{j,h=1}^2 b_{j,h} \xi_{\varepsilon}^j(y) \frac{\partial U_{\varepsilon, z_j, a_{\varepsilon, j}}}{\partial z_{j,h}}(y).$$

Let us remark that the previous system is solvable since, in view of the expansions (3.8)–(3.9), the elements of the matrix \mathbf{A} , $A_{i,j,h,k}$ for $i, j, h, k \in \{1, 2\}$, satisfy the following orthogonality properties:

$$\begin{aligned} A_{i,j,h,k} &= \int_{\Omega} \xi_{\varepsilon}^j \frac{\partial U_{\varepsilon, z_j, a_{\varepsilon, j}}}{\partial z_{j,h}} \frac{\partial V_{\varepsilon, Z, a_{\varepsilon, i}}}{\partial z_{i,k}} dy \\ &= \int_{B_{2s\varepsilon}(z_j)} \xi_{\varepsilon}^j \frac{\partial U_{\varepsilon, z_j, a_{\varepsilon, j}}}{\partial z_{j,h}} \frac{\partial U_{\varepsilon, Z, a_{\varepsilon, i}}}{\partial z_{i,k}} dy + O \left(\frac{\varepsilon}{|\ln \varepsilon|^2} \right) \\ &= c' \delta_{ij} \delta_{hk} \frac{1}{\left(\log \frac{R}{s\varepsilon} \right)^2} + O \left(\frac{\varepsilon}{|\ln \varepsilon|^2} \right), \end{aligned} \quad (4.3)$$

where

$$c' = \frac{\pi}{(s\phi_1'(s))^2} \left(\int_0^s (\phi_1'(t))^2 dt + \int_s^{2s} \frac{\xi(t)}{t^2} dt \right) > 0$$

is a constant.

With this notation, our main result in this section is the following.

Proposition 4.1. *For every fixed $Z = (z_1, z_2) \in \Omega \times \Omega$ satisfying (3.6) there exists $\varepsilon_0 > 0$ with the property that for all $0 < \varepsilon < \varepsilon_0$ there exists $\omega_{\varepsilon} \in E_{\varepsilon, Z}$ with*

$$\|\omega_{\varepsilon}\|_{L^{\infty}(\Omega)} \leq \|\omega_{\varepsilon}\|_{\infty} = O \left(\frac{\varepsilon}{|\ln \varepsilon|} \right)$$

and such that the function $u_{\varepsilon} := W_{\varepsilon} + \omega_{\varepsilon}$ is a solution of the “projected problem”:

$$Q_{\varepsilon} [\varepsilon^2 \Delta u_{\varepsilon} + (u_{\varepsilon} - 1)_+ - (-u_{\varepsilon} - \gamma)_+] = 0 \quad \text{in } \Omega. \quad (4.4)$$

In view of Proposition 4.1, the construction of a peak solution to (1.1) is reduced to finding $Z = (z_1, z_2)$ such that the solution u_ε to the projected problem (4.4) is actually a solution to the full problem (1.1).

The remaining part of this section is devoted to the proof of Proposition 4.1. We note that the linearization of the problem

$$-\varepsilon^2 \Delta u = (u - 1)_+ - (u + \gamma)_- \text{ in } \Omega, \quad u \in H_0^1(\Omega)$$

about a solution u is given by

$$-\varepsilon^2 \Delta \phi = \chi_{\{u > 1\} \cup \{u < -\gamma\}} \phi \text{ in } \Omega, \quad \phi \in H_0^1(\Omega).$$

Let $f_\varepsilon : \Omega \rightarrow \mathbb{R}$ be defined by

$$f_\varepsilon(y) = \begin{cases} 1, & \text{if } W_\varepsilon(y) - 1 > 0 \text{ or } W_\varepsilon(y) + \gamma < 0 \\ 0, & \text{otherwise.} \end{cases}$$

Remark 4.2. From Lemma 3.4 it follows that $f_\varepsilon(y) = 1$ in $B_{s\varepsilon(1-T\varepsilon)}(z_1) \cup B_{s\varepsilon(1-T\varepsilon)}(z_2)$ and $f_\varepsilon(y) = 0$ in $\Omega \setminus (B_{s\varepsilon(1+\varepsilon^\sigma)}(z_1) \cup B_{s\varepsilon(1+\varepsilon^\sigma)}(z_2))$.

We define $L_\varepsilon : W^{2,p}(\Omega) \cap H_0^1(\Omega) \rightarrow L^p(\Omega)$ by setting

$$L_\varepsilon u = -\varepsilon^2 \Delta u - f_\varepsilon(y)u.$$

Following the contradiction argument used in the proof of [10, Lemma 3.3], we have the following:

Lemma 4.3. Let $p > 1$ be fixed. There are constants $c_0 > 0$ and $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, Z satisfying (3.6), $u \in E_{\varepsilon,Z}$ with $Q_\varepsilon L_\varepsilon u = 0$ in $\Omega \setminus \cup_{j=1}^2 B_{L\varepsilon}(z_j)$ for some large $L > 0$, then

$$\|Q_\varepsilon L_\varepsilon u\|_{L^p(\Omega)} \geq c_0 \varepsilon^{2/p} \|u\|_{L^\infty(\Omega)}. \quad (4.5)$$

Consequently, $Q_\varepsilon L_\varepsilon : E_{\varepsilon,Z} \rightarrow F_{\varepsilon,Z}$ is one-to-one and onto, and for every $v \in F_{\varepsilon,Z}$ we have

$$\|(Q_\varepsilon L_\varepsilon u)^{-1} v\|_{L^\infty(\Omega)} \leq c_0^{-1} \varepsilon^{-2/p} \|v\|_{L^p(\Omega)}.$$

Proof. The proof follows [10] closely. We outline the main ideas for the reader's convenience. We first establish (4.5). Arguing by contradiction, we assume that there exist $\varepsilon_n \rightarrow 0$, $Z_n = (z_{1,n}, z_{2,n})$ satisfying (3.6) and $u_n \in E_{\varepsilon_n, Z_n}$, $\|u_n\|_\infty = 1$, such that $Q_{\varepsilon_n} L_{\varepsilon_n} u_n = 0$ in $\Omega \setminus \cup_{j=1}^2 B_{L\varepsilon_n}(z_{j,n})$ and $\|Q_{\varepsilon_n} L_{\varepsilon_n} u_n\|_p \leq \varepsilon_n^{2/p}/n$. Let $(\mathbf{b}_{1,n}, \mathbf{b}_{2,n}) \in \mathbb{R}^{2 \times 2}$ be such that

$$Q_{\varepsilon_n} L_{\varepsilon_n} u_n = L_{\varepsilon_n} u_n - \mathcal{B}(\mathbf{b}_{1,n}, \mathbf{b}_{2,n}).$$

where we have defined

$$\mathcal{B}(\mathbf{b}_{1,n}, \mathbf{b}_{2,n}) := \sum_{j,h=1}^2 b_{j,h,n} \xi \left(\frac{|y - z_{j,n}|}{s\varepsilon_n} \right) \frac{\partial U_{j,n}}{\partial z_{j,h}}$$

Then, similarly as in [10], using Lemma 3.4, (3.8) (3.9) and (4.3), we derive

$$b_{j,h,n} = O(\varepsilon_n^{1+\sigma} |\ln \varepsilon_n|).$$

Consequently, in virtue of (3.9), we have

$$\|\mathcal{B}(\mathbf{b}_{1,n}, \mathbf{b}_{2,n})\|_{L^p(\Omega)} = O\left(\varepsilon_n^{\frac{2}{p}+\sigma}\right)$$

and

$$\|L_{\varepsilon_n} u_n\|_{L^p(\Omega)} = \|Q_{\varepsilon_n} L_{\varepsilon_n} u_n\|_{L^p(\Omega)} + O(\varepsilon_n^{\frac{2}{p}+\sigma}) = o(\varepsilon_n^{\frac{2}{p}}).$$

Now, rescaling u_n around $z_{1,n}$, we define

$$\tilde{u}_{1,n}(y) := u_n(\varepsilon_n y + z_{1,n}), \quad y \in \Omega_n,$$

where $\Omega_n = \{y : \varepsilon_n y + z_{1,n} \in \Omega\}$. Since $\|\tilde{u}_{1,n}\|_\infty = \|u_n\|_\infty = 1$, we have that $\tilde{u}_{1,n}$ is bounded in $W_{\text{loc}}^{2,p}(\mathbb{R}^2)$ and there exists u_1 such that $\tilde{u}_{1,n} \rightarrow u_1$ in $C_{\text{loc}}^2(\mathbb{R}^2)$. On the other hand, in view of Lemma 3.4 we have that $f_{\varepsilon_n}(\varepsilon_n y + z_{1,n}) \rightarrow 1_{B_s(0)}$. We conclude that u_1 satisfies the problem

$$-\Delta u_1 - 1_{B_s(0)} u_1 = 0, \quad u_1 \in L^\infty(\mathbb{R}^2).$$

Now Proposition 3.1 in [10] implies that

$$u_1 = c_1 \frac{\partial U}{\partial x_1} + c_2 \frac{\partial U}{\partial x_2},$$

for some $c_1, c_2 \in \mathbb{R}$, where U is the basic cell function given by (2.1). On the other hand, taking limits in the orthogonality condition

$$\int_{\Omega} \Delta \left(\frac{\partial V_{\varepsilon_n, Z_n, i}}{\partial z_{i,n}} \right) u_n = 0,$$

which holds true since $u_n \in E_{\varepsilon_n, Z_n}$, we derive $c_1 = c_2 = 0$. We conclude that for any $L > 0$

$$\|u_n\|_{L^\infty(B_{L\varepsilon}(z_{1,n}))} = o(1).$$

By a similar rescaling argument at $z_{2,n}$ we also obtain that

$$\|u_n\|_{L^\infty(B_{L\varepsilon}(z_{2,n}))} = o(1).$$

Now, recalling that $Q_{\varepsilon_n} L_{\varepsilon_n} u_n = 0$ in $\Omega \setminus \bigcup_{j=1}^2 B_{L\varepsilon_n}(z_{j,n})$ we finally conclude that $\|u_n\|_\infty = o(1)$ which is a contradiction. Hence, (4.5) is established.

Now, we check that $Q_\varepsilon L_\varepsilon : E_{\varepsilon, Z} \rightarrow F_{\varepsilon, Z}$ is one-to-one and onto. Let $u \in E_{\varepsilon, Z}$ be such that $Q_\varepsilon L_\varepsilon u = 0$. Since the action of Q_ε is localized in $B_{2s\varepsilon}(z_1) \cup B_{2s\varepsilon}(z_2)$, we have $Q_\varepsilon L_\varepsilon u = L_\varepsilon u = 0$ in $\Omega \setminus (B_{L\varepsilon}(z_1) \cup B_{L\varepsilon}(z_2))$, for any $L \geq 2s$. Now estimate (4.5) yields $u = 0$ and the asserted one-to-one property follows.

We are left to check the surjectivity property. We note that by continuity of the projection operator P , we have $\frac{\partial V_j}{\partial z_{j,h}} = P \frac{\partial U_j}{\partial z_{j,h}} \in H_0^1(\Omega)$, $j, h = 1, 2$. In particular, for any $u \in E_{\varepsilon, Z}$ we have

$$\int_{\Omega} \Delta u \frac{\partial V_j}{\partial z_{j,h}} = \int_{\Omega} u \Delta \frac{\partial V_j}{\partial z_{j,h}} = 0, \quad j, h = 1, 2.$$

It follows that $Q_\varepsilon \Delta u = \Delta u$ for all $u \in E_{\varepsilon, Z}$ and it is easy to see that it is one to one and onto from $E_{\varepsilon, Z}$ to $F_{\varepsilon, Z}$. Now, let $v \in F_{\varepsilon, Z}$. We check that the equation $Q_\varepsilon L_\varepsilon u = v$ admits a solution in $E_{\varepsilon, Z}$. Equivalently, we seek a solution to

$$\varepsilon^2 u - (-Q_\varepsilon \Delta)^{-1} [Q_\varepsilon f_\varepsilon(y) u] = (-Q_\varepsilon \Delta)^{-1} v.$$

The operator on the r.h.s. above is of the form $\varepsilon^2 I + \text{compact operator}$. Then, by the Fredholm alternative, we conclude that $Q_\varepsilon L_\varepsilon : E_{\varepsilon, Z} \rightarrow F_{\varepsilon, Z}$ is onto. \square

We now define a fixed point problem which is equivalent to (4.4). We recall that the error term $\ell_\varepsilon = \ell_\varepsilon(x)$ is defined by

$$\begin{aligned} \ell_\varepsilon &:= \varepsilon^2 \Delta W_\varepsilon + (W_\varepsilon - 1)_+ - (-W_\varepsilon - \gamma)_+ \\ &= -(U_1 - a_1)_+ + (U_2 - a_2)_+ + (W_\varepsilon - 1)_+ - (-W_\varepsilon - \gamma)_+. \end{aligned}$$

We define the higher order error R_ε as follows:

$$R_\varepsilon(\omega) := (W_\varepsilon + \omega - 1)_+ - (-W_\varepsilon - \omega - \gamma)_+ - (W_\varepsilon - 1)_+ - (-W_\varepsilon - \gamma)_+ - f_\varepsilon(y)\omega.$$

We observe that if $\omega \in W^{2,p}(\Omega) \cap H_0^1(\Omega)$ satisfies

$$L_\varepsilon(\omega) = \ell_\varepsilon + R_\varepsilon(\omega)$$

then $u = W_\varepsilon + \omega$ is a solution to problem (1.1). We note that a solution ω to

$$Q_\varepsilon L_\varepsilon(\omega) = Q_\varepsilon(\ell_\varepsilon + R_\varepsilon(\omega)) \tag{4.6}$$

readily yields a solution to (4.4). In view of the invertibility property of $Q_\varepsilon L_\varepsilon$, as stated in Lemma 4.3, we define the operator $G_\varepsilon : E_{\varepsilon,Z} \rightarrow F_{\varepsilon,Z}$ by setting

$$G_\varepsilon(\omega) := (Q_\varepsilon L_\varepsilon)^{-1} Q_\varepsilon(\ell_\varepsilon + R_\varepsilon(\omega)).$$

Then, the projected problem (4.6) is equivalent to the fixed point problem

$$\omega = G_\varepsilon(\omega). \quad (4.7)$$

Arguing similarly as in the proof of [10, Proposition 3.6] one can prove that (4.7) has a unique solution. More precisely, we have the following Lemma which together with the previous observations provides the proof of Proposition 4.1.

Lemma 4.4. *There exists an $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and for any Z satisfying (3.6), equation (4.7) has a unique solution $\omega_\varepsilon \in E_{\varepsilon,Z}$ with*

$$\|\omega_\varepsilon\|_\infty = O\left(\frac{\varepsilon}{|\ln \varepsilon|}\right). \quad (4.8)$$

Proof. The proof follows [10] closely. We outline it for the reader's convenience. We define

$$M := E_{\varepsilon,Z} \cap \{\|\omega\|_\infty < \varepsilon\}.$$

We start by checking that G_ε is a map from M to M . Since the action of Q_ε is localized in $B_{2s\varepsilon}(z_1) \cup B_{2s\varepsilon}(z_1)$, we have

$$Q_\varepsilon(\ell_\varepsilon + R_\varepsilon(\omega)) = \ell_\varepsilon + R_\varepsilon(\omega) = 0 \quad \text{in } \Omega \setminus (B_{L\varepsilon}(z_1) \cup B_{L\varepsilon}(z_1)),$$

for $L \geq 2s$. Therefore, we may use estimate (4.5) to obtain

$$\|G_\varepsilon(\omega)\|_\infty = \|(Q_\varepsilon L_\varepsilon)^{-1}(Q_\varepsilon \ell_\varepsilon + Q_\varepsilon R_\varepsilon(\omega))\|_\infty \leq C\varepsilon^{-2/p} \|Q_\varepsilon \ell_\varepsilon + Q_\varepsilon R_\varepsilon(\omega)\|_p.$$

Similarly as in [10], we estimate

$$\begin{aligned} \|Q_\varepsilon \ell_\varepsilon + Q_\varepsilon R_\varepsilon(\omega)\|_p &\leq C(\|\ell_\varepsilon\|_p + \|R_\varepsilon(\omega)\|_p) \\ \|\ell_\varepsilon\|_p &\leq \frac{C\varepsilon^{2/p+1}}{|\ln \varepsilon|} \\ \|R_\varepsilon(\omega)\|_p &\leq C\varepsilon^{(2+\sigma)/p} \|\omega\|_\infty. \end{aligned}$$

Since $\omega \in M$, we have $\|\omega\|_\infty \leq \varepsilon$ and we conclude that

$$\|G_\varepsilon(\omega)\|_\infty \leq C\varepsilon^{-2/p} \left(\frac{\varepsilon^{2/p+1}}{|\ln \varepsilon|} + \varepsilon^{(2+\sigma)/p} \|\omega\|_\infty \right) \leq \frac{C\varepsilon}{|\ln \varepsilon|}. \quad (4.9)$$

In particular, G_ε maps M into M .

By similar arguments, we can also check the contraction property:

$$\|G_\varepsilon(\omega_1) - G_\varepsilon(\omega_2)\|_\infty \leq C\varepsilon^{-2/p} \|R_\varepsilon(\omega_1) - R_\varepsilon(\omega_2)\|_p = o(1) \|\omega_1 - \omega_2\|_\infty.$$

It follows that for all sufficiently small $\varepsilon > 0$ there exists a fixed point $\omega_\varepsilon \in M$ for G_ε . Moreover, in view of (4.9), we obtain the asserted estimate

$$\|\omega_\varepsilon\|_\infty = \|G_\varepsilon(\omega_\varepsilon)\|_\infty \leq \frac{C\varepsilon}{|\ln \varepsilon|}.$$

□

Proof of Proposition 4.1. Now the proof follows as direct consequence of Lemma 4.3 and Lemma 4.4. □

5. FREE BOUNDARY PROPERTIES FOR THE PROJECTED PROBLEM

In order to show that $(z_1, z_2) \rightarrow \omega_\varepsilon$ is C^1 it is essential to show that the free boundaries $\{u_\varepsilon = 1\}$ and $\{u_\varepsilon = -\gamma\}$ have two-dimensional Lebesgue measure equal to zero, see the *Step 2* in the proof of Proposition 6.2 below. The aim of this section is to establish such a property, via an argument involving the Faber-Krahn inequality. Throughout this section, for any $S \subset \mathbb{R}^2$, we denote by mS the two-dimensional Lebesgue measure of S . The main result in this section is the following.

Proposition 5.1. *Let u_ε be the solution to the “projected problem” (4.4), as obtained in Proposition 4.1. It holds that*

$$m\{u_\varepsilon = 1\} = 0 = m\{u_\varepsilon = -\gamma\}.$$

Before proving the previous result, we observe that arguing exactly as in Lemma 3.4, in view of the decay estimate (4.8) for ω_ε , we can prove the following useful level set estimates.

Lemma 5.2. *There exist $0 < \varepsilon_0 \ll 1$ and $T \gg 1$ such that for all $0 < \varepsilon < \varepsilon_0$ we have*

$$\begin{aligned} W_\varepsilon(y) + \omega_\varepsilon - 1 &> 0, & y \in B_{s\varepsilon(1-T\varepsilon)}(z_1), \\ -W_\varepsilon(y) - \omega_\varepsilon - \gamma &> 0, & y \in B_{s\varepsilon(1-T\varepsilon)}(z_2) \\ -\gamma < W_\varepsilon(y) + \omega_\varepsilon &< 1, & y \in \Omega \setminus (B_{s\varepsilon(1+\varepsilon^\sigma)}(z_1) \cup B_{s\varepsilon(1+\varepsilon^\sigma)}(z_2)). \end{aligned}$$

where $\sigma > 0$ is a small constant.

Proof of Proposition 5.1. Being a solution to the projected problem (4.4), u_ε satisfies

$$-\varepsilon^2 \Delta u_\varepsilon = (u_\varepsilon - 1)_+ - (-u_\varepsilon - \gamma)_+ - \xi_\varepsilon^1(y) \langle \mathbf{b}_1, \nabla_{z_1} U_1(y) \rangle - \xi_\varepsilon^2(y) \langle \mathbf{b}_2, \nabla_{z_2} U_2(y) \rangle$$

for some $\mathbf{b}_1, \mathbf{b}_2 \in \mathbb{R}^2$.

We first show that $m\{u_\varepsilon = 1\} = 0$. Arguing by contradiction, suppose that $m\{u_\varepsilon = 1\} > 0$. In view of Lemma 5.2 we have $\{u_\varepsilon = 1\} \subset B_{2s\varepsilon}(z_1)$. In $\{u_\varepsilon = 1\} \subset B_{2s\varepsilon}(z_1)$ we have $\Delta u_\varepsilon \equiv 0$ and therefore u_ε satisfies, in $\{u_\varepsilon = 1\}$

$$0 = -\varepsilon^2 \Delta u_\varepsilon = (u_\varepsilon - 1)_+ - \langle \mathbf{b}_1, \nabla_{z_1} U_1(y) \rangle = -\langle \mathbf{b}_1, \nabla_{z_1} U_1(y) \rangle$$

Since $\partial U_1 / \partial z_{1,1}$ and $\partial U_1 / \partial z_{1,2}$ are linearly independent, we conclude that $\mathbf{b}_1 = 0$. In particular, u_ε satisfies

$$-\varepsilon^2 \Delta u_\varepsilon = (u_\varepsilon - 1)_+ \text{ in } B_{2s\varepsilon}(z_1).$$

Let

$$\Omega_{\varepsilon,1} := \{u_\varepsilon > 1\}.$$

Claim. $\Omega_{\varepsilon,1}$ has exactly one connected component.

Indeed, suppose the contrary and let $\tilde{\omega}_\varepsilon \subset \Omega_{\varepsilon,1}$ be a connected component of $\Omega_{\varepsilon,1}$ with $z_1 \notin \tilde{\omega}_\varepsilon$. Note that $\tilde{\omega}_\varepsilon$ is an open subset of Ω . Then, u_ε satisfies the problem

$$\begin{cases} -\varepsilon^2 \Delta u_\varepsilon = u_\varepsilon - 1 & \text{in } \tilde{\omega}_\varepsilon \\ u_\varepsilon = 1 & \text{on } \partial \tilde{\omega}_\varepsilon. \end{cases}$$

Setting $v_\varepsilon = u_\varepsilon - 1$, we derive

$$\begin{cases} -\varepsilon^2 \Delta v_\varepsilon = v_\varepsilon & \text{in } \tilde{\omega}_\varepsilon \\ v_\varepsilon = 0 & \text{on } \partial \tilde{\omega}_\varepsilon. \end{cases}$$

Multiplying by v_ε and integrating, we obtain

$$\varepsilon^2 \int_{\tilde{\omega}_\varepsilon} |\nabla v_\varepsilon|^2 = \int_{\tilde{\omega}_\varepsilon} v_\varepsilon^2 \leq \frac{1}{\lambda_1(\tilde{\omega}_\varepsilon)} \int_{\tilde{\omega}_\varepsilon} |\nabla v_\varepsilon|^2$$

and consequently

$$\lambda_1(\tilde{\omega}_\varepsilon) \leq \frac{1}{\varepsilon^2}.$$

On the other hand, setting $A_\varepsilon := B_{s\varepsilon(1+\varepsilon^\sigma)} \setminus B_{s\varepsilon(1-T\varepsilon)}$, we note, again by Lemma 5.2, that $\tilde{\omega}_\varepsilon \subset A_\varepsilon$ and $mA_\varepsilon = 2\pi s^2 \varepsilon^{2+\sigma}(1+o(1))$ as $\varepsilon \rightarrow 0$. It follows that

$$\lambda_1(\tilde{\omega}_\varepsilon) \geq \lambda_1(A_\varepsilon) \geq \lambda_1(A_\varepsilon^*) = \frac{\pi j_0^2}{2\pi s^2 \varepsilon^{2+\sigma}(1+o(1))}$$

where A_ε^* denotes a ball with measure $mA_\varepsilon^* = mA_\varepsilon$, and where we used the Faber-Krahn inequality (see, e.g., [2]) to derive the last inequality. This is a contradiction. Hence, $\Omega_{\varepsilon,1}$ is connected and the claim is established. Arguing in a similar way we can prove that the set $\Omega_{\varepsilon,-\gamma}^- := \{u_\varepsilon < -\gamma\}$ is connected. Finally, we claim that by virtue of Lemma 5.2, the maximum principle implies that the set $\Omega_{\varepsilon,-\gamma}^1 := \{x \in \Omega : -\gamma < u_\varepsilon(x) < 1\}$ is also connected. Indeed, arguing by contradiction we suppose that there exists a connected component $\hat{\omega}_\varepsilon$ of $\Omega_{\varepsilon,-\gamma}^1$ with $\hat{\omega}_\varepsilon \subset B_{2s\varepsilon}(z_1)$. Then u_ε is a solution of

$$\begin{cases} \Delta u_\varepsilon = 0 & \text{in } \hat{\omega}_\varepsilon \\ u_\varepsilon = 1 & \text{on } \partial \hat{\omega}_\varepsilon \\ u_\varepsilon < 1 & \text{in } \hat{\omega}_\varepsilon, \end{cases}$$

a contradiction.

Now we can apply Theorem 1 and Theorem 2 in [16] together with the argument in [17], p. 136, to conclude that $\nabla u_\varepsilon \neq 0$ on $\{u_\varepsilon = 1\}$. Therefore, $\{u_\varepsilon = 1\}$ is a simple C^2 -curve. In particular $m\{u_\varepsilon = 1\} = 0$, a contradiction.

The remaining part of the statement is obtained similarly. \square

6. THE REDUCED FUNCTIONAL AND THE PROOF OF THEOREM 2.1

We recall that the Euler-Lagrange functional for (1.1) is given by

$$I_\varepsilon(u) = \frac{\varepsilon^2}{2} \int_\Omega |\nabla u|^2 - \frac{1}{2} \int_\Omega [(u-1)_+]^2 - \frac{1}{2} \int_\Omega [(-u-\gamma)_+]^2, \quad u \in H_0^1(\Omega).$$

For every $z_1, z_2 \in \Omega$ satisfying (3.6) we define the “reduced functional”

$$K_\varepsilon(z_1, z_2) = I_\varepsilon(W_\varepsilon + \omega_\varepsilon),$$

where $\omega_\varepsilon = \omega_\varepsilon(Z)$ is the error function obtained in Proposition 4.1. Then, we have:

Lemma 6.1. *If $Z = (z_1, z_2)$ is a critical point for K_ε , then u_ε defined by*

$$u_\varepsilon = PU_{\varepsilon, a_{\varepsilon,1}(Z), z_1} - PU_{\varepsilon, a_{\varepsilon,2}(Z), z_2} + \omega_\varepsilon$$

is a solution for problem (1.1).

Proof. By construction, $u_\varepsilon = W_\varepsilon + \omega_\varepsilon$ satisfies the “projected problem”

$$-\varepsilon^2 \Delta u_\varepsilon - (u_\varepsilon - 1)_+ + (-u_\varepsilon - \gamma)_+ = \sum_{j,h=1}^2 b_{jh} \xi \left(\frac{|y - z_j|}{s\varepsilon} \right) \frac{\partial U_j}{\partial z_{j,h}}, \quad (6.1)$$

for some constants b_{jh} , $j, h = 1, 2$. Equivalently, we have

$$\langle I'_\varepsilon(u_\varepsilon), \varphi \rangle = \sum_{j,h=1}^2 b_{jh} \int_{B_{2s\varepsilon}(z_j)} \xi \left(\frac{|y - z_j|}{s\varepsilon} \right) \frac{\partial U_j}{\partial z_{j,h}} \varphi,$$

for all $\varphi \in H_0^1(\Omega)$. We verify that if (z_1, z_2) is a critical point of K_ε , then the constants b_{jh} in (6.1) vanish for all $j, h = 1, 2$. Indeed, we have:

$$\frac{\partial K_\varepsilon(Z)}{\partial z_{j,h}} = \langle I'(u_\varepsilon), \frac{\partial u_\varepsilon}{\partial z_{j,h}} \rangle = \sum_{i,\bar{h}=1}^2 b_{i,\bar{h}} \int_{B_{2s\varepsilon}(z_i)} \xi \left(\frac{|y - z_i|}{s\varepsilon} \right) \frac{\partial U_i}{\partial z_{i,\bar{h}}} \left(\frac{\partial PU_1}{\partial z_{j,h}} - \frac{\partial PU_2}{\partial z_{j,h}} + \frac{\partial \omega_\varepsilon}{\partial z_{j,h}} \right).$$

In view of (3.9) we conclude that $b_{jh} = 0$, $j, h = 1, 2$. \square

Hence, in order to conclude the proof of Theorem 2.1 we are left to obtain a critical point for K_ε . For $a_1, a_2 > 0$ let \mathcal{H}_{a_1, a_2} be the Kirchhoff-Routh type Hamiltonian defined by

$$\mathcal{H}_{a_1, a_2}(z_1, z_2) = a_1^2 h(z_1) + 2a_1 a_2 \bar{G}(z_1, z_2) + a_2^2 h(z_2).$$

Recall that

$$\mathcal{M} = \{(z_1, z_2) \in \Omega \times \Omega : z_1 \neq z_2\}.$$

The following proposition clarifies the relation between K_ε and \mathcal{H}_{a_1, a_2} :

Proposition 6.2. *The following expansions hold true as $\varepsilon \rightarrow 0$:*

$$\begin{aligned} K_\varepsilon(z_1, z_2) &= I_\varepsilon(W_\varepsilon) + O\left(\frac{\varepsilon^3}{|\ln \varepsilon|}\right) \\ \frac{\partial K_\varepsilon(z_1, z_2)}{\partial z_{i,h}} &= \frac{\partial I_\varepsilon(W_\varepsilon)}{\partial z_{i,h}} + O\left(\frac{\varepsilon^{2+\sigma}}{|\ln \varepsilon|}\right), \end{aligned} \quad (6.2)$$

uniformly on compact subsets of \mathcal{M} . Furthermore,

$$I_\varepsilon(W_\varepsilon) = -A_1 \frac{\varepsilon^2 k_\varepsilon}{\ln \frac{R}{s\varepsilon}} \mathcal{H}_{a_{\varepsilon,1}, a_{\varepsilon,2}}(z_1, z_2) + \varepsilon^2 k_\varepsilon A_{2,\varepsilon} - \varepsilon^2 k_\varepsilon^2 A_{3,\varepsilon} + O\left(\frac{\varepsilon^3}{|\ln \varepsilon|}\right), \quad (6.3)$$

where $A_1, A_{2,\varepsilon}, A_{3,\varepsilon} > 0$ are the uniformly bounded constants defined by

$$\begin{aligned} A_1 &= \frac{1}{2} \int_{B_s(0)} \varphi_1, \\ A_{2,\varepsilon} &= \frac{a_{\varepsilon,1}^2 + a_{\varepsilon,2}^2}{2} \int_{B_s(0)} \varphi_1, \\ A_{3,\varepsilon} &= \frac{a_{\varepsilon,1}^2 + a_{\varepsilon,2}^2}{2} \int_{B_s(0)} \varphi_1^2. \end{aligned} \quad (6.4)$$

In order to prove Proposition 6.2 we will use the following lemmata.

Lemma 6.3. *Let $(z, \zeta) \in \mathcal{M}$. For any $a, b > 0$ and for any sufficiently small $\varepsilon > 0$ the following expansions hold:*

$$\begin{aligned} \text{(i)} \quad \int_{\Omega} |\nabla P U_{\varepsilon, a, z}(x)|^2 dx &= a^2 k_\varepsilon \left[\int_{B_s(0)} \varphi_1 + k_\varepsilon \int_{B_s(0)} \varphi_1^2 - \frac{\int_{B_s(0)} \varphi_1}{\ln \frac{R}{s\varepsilon}} g(z, z) \right] + O\left(\frac{\varepsilon}{|\ln \varepsilon|^2}\right), \\ \text{(ii)} \quad \int_{\Omega} \nabla P U_{\varepsilon, a, z}(x) \cdot \nabla P U_{\varepsilon, b, \zeta}(x) dx &= ab k_\varepsilon \frac{\int_{B_s(0)} \varphi_1}{\ln \frac{R}{s\varepsilon}} \bar{G}(z, \zeta) + O\left(\frac{\varepsilon}{|\ln \varepsilon|^2}\right), \end{aligned}$$

uniformly on compact subsets of \mathcal{M} .

Proof. Proof of (i). We compute, recalling (3.2) and Lemma 3.1:

$$\begin{aligned} \int_{\Omega} |\nabla P U_{\varepsilon, a, z}(x)|^2 dx &= - \int_{\Omega} (\Delta P U_{\varepsilon, a, z})(x) P U_{\varepsilon, a, z}(x) dx = \frac{1}{\varepsilon^2} \int_{B_{s\varepsilon}(z)} (U_{\varepsilon, a, z} - a)_+ P U_{\varepsilon, a, z} dx \\ &= \frac{1}{\varepsilon^2} \int_{B_{s\varepsilon}(z)} a^2 k_\varepsilon \varphi_1 \left(\frac{x - z}{\varepsilon} \right) \left[1 + k_\varepsilon \varphi_1 \left(\frac{x - z}{\varepsilon} \right) - \frac{g(x, z)}{\ln \frac{R}{s\varepsilon}} \right] dx \\ &= a^2 k_\varepsilon \int_{B_s(0)} \varphi_1(y) \left[1 + k_\varepsilon \varphi_1(y) - \frac{g(z + \varepsilon y, z)}{\ln \frac{R}{s\varepsilon}} \right] dy \\ &= a^2 k_\varepsilon \int_{B_s(0)} \varphi_1(y) \left[1 + k_\varepsilon \varphi_1(y) - \frac{g(z, z)}{\ln \frac{R}{s\varepsilon}} + O\left(\frac{\varepsilon}{|\ln \varepsilon|}\right) \right] dy. \end{aligned}$$

Proof of (ii). Similarly, we compute:

$$\begin{aligned}
\int_{\Omega} \nabla PU_{\varepsilon,a,z}(x) \cdot \nabla PU_{\varepsilon,b,\zeta}(x) dx &= - \int_{\Omega} (\Delta PU_{\varepsilon,a,z})(x) PU_{\varepsilon,b,\zeta}(x) dx \\
&= \frac{1}{\varepsilon^2} \int_{B_{s\varepsilon}(z)} (U_{\varepsilon,a,z} - a)_+ PU_{\varepsilon,b,\zeta} dx = \frac{1}{\varepsilon^2} \int_{B_{s\varepsilon}(z)} ak_{\varepsilon} \varphi_1 \left(\frac{x-z}{\varepsilon} \right) \frac{b}{\ln \frac{R}{s\varepsilon}} \bar{G}(x, \zeta) dx \\
&= \frac{ab k_{\varepsilon}}{\ln \frac{R}{s\varepsilon}} \int_{B_s(0)} \varphi_1(y) \bar{G}(z + \varepsilon y, \zeta) dy = \frac{ab k_{\varepsilon}}{\ln \frac{R}{s\varepsilon}} \int_{B_s(0)} \varphi_1(y) (\bar{G}(z, \zeta) + O(\varepsilon|y|)) dy \\
&= \frac{ab k_{\varepsilon}}{\ln \frac{R}{s\varepsilon}} \int_{B_s(0)} \varphi_1(y) \bar{G}(z, \zeta) dy + O \left(\frac{\varepsilon}{|\ln \varepsilon|^2} \right).
\end{aligned}$$

□

Lemma 6.4. *The following expansions hold:*

- (i) $\int_{\Omega} [(W_{\varepsilon} - 1)_+]^2 dx = \varepsilon^2 a_1^2 k_{\varepsilon}^2 \int_{B_s(0)} \varphi_1^2 + O \left(\frac{\varepsilon^3}{|\ln \varepsilon|} \right);$
- (ii) $\int_{\Omega} [(-W_{\varepsilon} - \gamma)_+]^2 dx = \varepsilon^2 a_2^2 k_{\varepsilon}^2 \int_{B_s(0)} \varphi_1^2 + O \left(\frac{\varepsilon^3}{|\ln \varepsilon|} \right).$

The convergences are uniform on compact subsets of \mathcal{M} .

Proof. We write:

$$\begin{aligned}
\int_{\Omega} [(W_{\varepsilon} - 1)_+]^2 dx &= \int_{B_{s\varepsilon}(z_1)} [(W_{\varepsilon} - 1)_+]^2 dx + \int_{B_{s\varepsilon}(z_2)} [(W_{\varepsilon} - 1)_+]^2 dx \\
&\quad + \int_{\Omega \setminus (B_{s\varepsilon}(z_1) \cup B_{s\varepsilon}(z_2))} [(W_{\varepsilon} - 1)_+]^2 dx.
\end{aligned}$$

In view of Lemma 3.2 we compute:

$$\begin{aligned}
\int_{B_{s\varepsilon}(z_1)} [(W_{\varepsilon} - 1)_+]^2 dx &= \int_{B_{s\varepsilon}(z_1)} \left[a_1 k_{\varepsilon} \varphi_1 \left(\frac{x - z_1}{\varepsilon} \right) + O \left(\frac{|x - z_1|}{|\ln \varepsilon|} \right) \right]^2 dx \\
&= a_1^2 k_{\varepsilon}^2 \int_{B_{s\varepsilon}(z_1)} \varphi_1^2 \left(\frac{x - z_1}{\varepsilon} \right) dx + O \left(\frac{\varepsilon^3}{|\ln \varepsilon|^2} \right) \\
&= \varepsilon^2 a_1^2 k_{\varepsilon}^2 \int_{B_s(0)} \varphi_1^2 + O \left(\frac{\varepsilon^3}{|\ln \varepsilon|^2} \right).
\end{aligned}$$

On the other hand, for sufficiently small $\varepsilon > 0$ we have

$$\int_{B_{s\varepsilon}(z_2)} [(W_{\varepsilon} - 1)_+]^2 dx = 0 = \int_{\Omega \setminus (B_{s\varepsilon}(z_1) \cup B_{s\varepsilon}(z_2))} [(W_{\varepsilon} - 1)_+]^2 dx.$$

The remaining estimates are derived similarly. □

Proof of Proposition 6.2. In view of Lemma 6.3 we have

$$\begin{aligned}
\int_{\Omega} |\nabla W_{\varepsilon}|^2 dx &= \int_{\Omega} (|\nabla PU_1|^2 - 2 \nabla PU_1 \cdot \nabla PU_2 + |\nabla PU_1|^2) dx \\
&= \frac{k_{\varepsilon}}{\ln \frac{R}{s\varepsilon}} \int_{B_s(0)} \varphi_1 [-a_{\varepsilon,1}^2 g(z_1, z_1) - 2a_{\varepsilon,1} a_{\varepsilon,2} \bar{G}(z_1, z_2) - a_{\varepsilon,2}^2 g(z_2, z_2)] \\
&\quad + (a_{\varepsilon,1}^2 + a_{\varepsilon,2}^2) k_{\varepsilon} \left(\int_{B_s(0)} \varphi_1 + k_{\varepsilon} \int_{B_s(0)} \varphi_1^2 \right) + O \left(\frac{\varepsilon}{|\ln \varepsilon|^2} \right). \tag{6.5}
\end{aligned}$$

We conclude the proof if the first part of (6.2) in view of Lemma 6.4.

We are left to prove the second part of (6.2), namely the C^1 -approximation property. The proof closely follows [10, Section 3] therefore we only outline the main steps.

Step 1. We first observe that $\omega_{\varepsilon,Z}$ is continuous in H^1 -norm. We already know that $\omega_{\varepsilon,Z}$ is uniformly bounded in $L^\infty(\Omega)$, uniformly in Z satisfying (3.6). The asserted continuity property follows by elliptic regularity applied to the projected problem

$$-\varepsilon^2 \Delta(W_\varepsilon + \omega_\varepsilon) - (W_\varepsilon + \omega_\varepsilon - 1)_+ - (-W_\varepsilon - \omega_\varepsilon - \gamma)_+ = \sum_{j,h=1}^2 b_{j,h} \xi \left(\frac{|y - z_j|}{s\varepsilon} \right) \frac{\partial U_j}{\partial z_{j,h}}.$$

Step 2. We prove that $\frac{\partial \omega_{\varepsilon,Z}}{\partial z_{j,h}}$ is continuous in H^1 -norm and

$$\left\| \frac{\partial \omega_{\varepsilon,Z}}{\partial z_{j,h}} \right\|_\infty = O \left(\frac{1}{\varepsilon^{1-\sigma} |\ln \varepsilon|} \right).$$

The proof is based on a differential quotients techniques and strongly relies on Proposition 5.1. More precisely, let $\mathbf{e} \in \mathbb{R}^4$ be a unit vector and let $s \neq 0$, for any function v_Z we denote the s -difference quotient of v_Z in the direction \mathbf{e} at Z with

$$\Delta^s v_Z(y) = \frac{v_{Z+s\mathbf{e}}(y) - v_Z(y)}{s}.$$

For any $a \in \mathbb{R}$, let $K_s^a := \{y : d(y, \{u = a\}) < s\}$. Then,

$$\Delta^s(u - a)_+ = 1_{\{u > a\}} \Delta^s u + O(1_{K_s^a} |\Delta^s u|).$$

Now the crucial consequence of Proposition 5.1 is that $mK_s^1 \rightarrow 0$ and $mK_s^{-\gamma} \rightarrow 0$ as $s \rightarrow 0$. A careful analysis as in the proof of [10, Lemma 3.7] together with elliptic regularity applied to $\Delta^s \omega_\varepsilon$ yields the desired property

$$\|D_{\mathbf{e}} \omega_{\varepsilon,Z}\|_\infty \leq \frac{C}{\varepsilon^{1-\sigma} |\ln \varepsilon|}.$$

□

Proof of Theorem 2.1. We argue as in the proof of Theorem 1.3 in [15]. we equivalently seek critical points for the functional F_ε defined by

$$F_\varepsilon(z_1, z_2) = -\frac{\ln \frac{R}{s\varepsilon}}{\varepsilon^2 k_\varepsilon} A_1^{-1} (K_\varepsilon(z_1, z_2) - \varepsilon^2 k_\varepsilon A_{2,\varepsilon} + \varepsilon^2 k_\varepsilon^2 A_{3,\varepsilon}),$$

where A_1 , $A_{2,\varepsilon}$ and $A_{3,\varepsilon}$ are the constants defined in (6.4). In view of Proposition 6.2, F_ε is approximated by $\mathcal{H}_{a_{\varepsilon,1}, a_{\varepsilon,2}}$, uniformly in C^1 on compact subsets of \mathcal{M} . Moreover, $\mathcal{H}_{a_{\varepsilon,1}, a_{\varepsilon,2}}(z_1, z_2) \rightarrow +\infty$ as $(z_1, z_2) \rightarrow \partial\mathcal{M}$. Let $C \subset \mathcal{M}$ be a compact set such that $\text{cat} C = \text{cat} \mathcal{M}$. Let $\mathcal{U} \subset \mathcal{M}$ be an open set such that $C \subset \mathcal{U}$ and

$$\inf_{\partial\mathcal{U}} \mathcal{H}_{a_{\varepsilon,1}, a_{\varepsilon,2}} > \max_C \mathcal{H}_{a_{\varepsilon,1}, a_{\varepsilon,2}}.$$

By taking $\varepsilon > 0$ sufficiently small, we derive

$$\inf_{\partial\mathcal{U}} F_\varepsilon > \max_C F_\varepsilon.$$

For $j = 1, 2, \dots, \text{cat} \mathcal{M}$ we set

$$\begin{aligned} c_\varepsilon^j &:= \inf \{c : \text{cat}_{\mathcal{M}}(F_\varepsilon^c \cap \mathcal{U}) \geq j\} \\ &= \inf \{\max F_\varepsilon : A \subset \mathcal{U} \text{ compact, } \text{cat}_{\mathcal{M}} A \geq j\}, \end{aligned}$$

where $F_\varepsilon^c = \{(z_1, z_2) \in \mathcal{M} : F_\varepsilon(z_1, z_2) \leq c\}$. Now, standard arguments based on the Deformation Lemma (see, e.g., [1], Theorem 2.3) imply that c_ε^j , $j = 1, 2, \dots, \text{cat} \mathcal{M}$, are critical levels for F_ε . Finally, we observe that

$$c_\varepsilon^1 = \min_{\mathcal{M}} F_\varepsilon.$$

See also Theorem 2.1 in [5] and Theorem 1.1 in [19].

Proof of (i). The proof follows by analogous arguments as in Proposition 5.1.

Proof of (ii). If $\gamma = 1$, for every solution u_ε to (1.1) we obtain a second solution given by $-u_\varepsilon$. Hence, the result follows by standard symmetry arguments. \square

7. QUALITATIVE PROPERTIES OF u_ε AND PROOF OF THEOREM 2.2

We recall that the Kirchhoff-Routh type Hamiltonian \mathcal{H}_γ is defined by:

$$\mathcal{H}_\gamma(z_1, z_2) = h(z_1) + 2\gamma\bar{G}(z_1, z_2) + \gamma^2 h(z_2)$$

for all $(z_1, z_2) \in \mathcal{M}$. Let $(z_1^\gamma, z_2^\gamma) \in \mathcal{M}$ be a minimum point for \mathcal{H}_γ , namely

$$\mathcal{H}_\gamma(z_1^\gamma, z_2^\gamma) = \min_{\mathcal{M}} \mathcal{H}_\gamma.$$

We note that since $h(z) \rightarrow +\infty$ as $z \rightarrow \partial\Omega$ and $\bar{G}(z_1, z_2) \rightarrow +\infty$ as $d(z_1, z_2) \rightarrow 0$, the minimum of \mathcal{H}_γ is well-defined and it is attained in \mathcal{M} .

We recall that $\underline{z} \in \Omega$ is a *harmonic center* for Ω if it is a minimum point for h , that is:

$$h(\underline{z}) = \min_{\Omega} h \geq 0.$$

For every $\gamma \in (0, 1)$ let (z_1^γ, z_2^γ) be a minimum point for \mathcal{H}_γ , that is:

$$\mathcal{H}_\gamma(z_1^\gamma, z_2^\gamma) = \min_{\mathcal{M}} \mathcal{H}_\gamma.$$

Up to passing to a subsequence, we may assume that $(z_1^\gamma, z_2^\gamma) \rightarrow (z_1^0, z_2^0) \in \bar{\Omega} \times \bar{\Omega}$ as $\gamma \rightarrow 0^+$. The following proposition contains the main ingredients needed for the proof of Theorem 2.2.

Proposition 7.1. *The following results hold true.*

- (i) $z_1^0 \in \Omega$; moreover, z_1^0 is a harmonic center for Ω .
- (ii) $z_2^0 \in \partial\Omega$; moreover $\partial_\nu(z_1^0, z_2^0) = \max_{p \in \partial\Omega} G_\nu(z_1^0, p)$, where ν denotes the outer normal direction to $\partial\Omega$.

Proof. We begin by proving that for every $\eta > 0$ there exist a harmonic center $\underline{z} \in \Omega$ and $0 < \gamma_\eta \ll 1$ such that

$$d(z_1^\gamma, \underline{z}) < \eta, \quad d(z_2^\gamma, \partial\Omega) < \eta.$$

To this end, we first check that there exists $M > 0$ independent of γ such that

$$\min_{\mathcal{M}} \mathcal{H}_\gamma \leq M. \tag{7.1}$$

Indeed, fix any $(\bar{z}_1, \bar{z}_2) \in \mathcal{M}$. Then, taking into account of (2.4),

$$\begin{aligned} \min_{\mathcal{M}} \mathcal{H}_\gamma &\leq \mathcal{H}_\gamma(\bar{z}_1, \bar{z}_2) = h(\bar{z}_1) + 2\gamma\bar{G}(\bar{z}_1, \bar{z}_2) + \gamma^2 h(\bar{z}_2) \\ &\leq h(\bar{z}_1) + 2\bar{G}(\bar{z}_1, \bar{z}_2) + h(\bar{z}_2) =: M. \end{aligned}$$

Claim 1. $z_1^0 \notin \partial\Omega$.

Arguing by contradiction, we assume that $z_1^\gamma \rightarrow \partial\Omega$ as $\gamma \rightarrow 0^+$. We deduce that

$$\begin{aligned} \min_{\mathcal{M}} \mathcal{H}_\gamma &= \mathcal{H}_\gamma(z_1^\gamma, z_2^\gamma) = h(z_1^\gamma) + 2\gamma\bar{G}(z_1^\gamma, z_2^\gamma) + \gamma^2 h(z_2^\gamma) \\ &\geq h(z_1^\gamma) \rightarrow +\infty, \end{aligned}$$

a contradiction to (7.1). We conclude that $z_1^0 \in \Omega$.

Claim 2. $z_2^0 \neq z_1^0$.

Arguing by contradiction, assume that $d(z_1^\gamma, z_2^\gamma) \rightarrow 0$ as $\gamma \rightarrow 0^+$. We deduce that

$$\begin{aligned} \min_{\mathcal{M}} \mathcal{H}_\gamma &= \mathcal{H}_\gamma(z_1^\gamma, z_2^\gamma) \geq h(z_1^\gamma) + 2\gamma\bar{G}(z_1^\gamma, z_2^\gamma) + \min_{\Omega} h \\ &\geq h(z_1^0) + 2\gamma\bar{G}(z_1^\gamma, z_2^\gamma) + \min_{\Omega} h - 1 \rightarrow +\infty, \end{aligned}$$

a contradiction to (7.1).

Claim 3. $d(z_2^\gamma, \partial\Omega) \rightarrow 0$.

Arguing by contradiction and we assume that there exists $\varepsilon_0 > 0$ such that $d(z_2^\gamma, \partial\Omega) \geq 2\varepsilon_0$ for all $\gamma \in (0, \gamma'_\eta)$. For every $x \in \Omega$ let $d_x = d(x, \partial\Omega)$. Without loss of generality, we may assume that the tubular neighborhood

$$\Omega_{2\varepsilon_0} = \{x \in \Omega : d_x < 2\varepsilon_0\}$$

is well-defined. We denote by $\tilde{\pi}_{\varepsilon_0}$ the projection $\tilde{\pi}_{\varepsilon_0} : \Omega_{2\varepsilon_0} \rightarrow \partial\Omega_{\varepsilon_0} \setminus \partial\Omega$. In view of the Hopf maximum principle we may also assume that for any $\zeta \in \partial\Omega$ the function

$$\psi(t) := \bar{G}(z_1^\gamma, \zeta - t\nu_\zeta),$$

where ν_ζ denotes the outer normal at ζ , is strictly increasing for $t \in [0, 2\varepsilon_0]$. Moreover, applying the maximum principle to the harmonic function $\bar{G}(z_1^\gamma, z_2^\gamma)$ in $(\Omega \setminus B_\rho(z_1^\gamma)) \setminus \Omega_{2\varepsilon_0}$, where $\rho > 0$ is a small constant, we find $\tilde{z}_2^\gamma \in \partial\Omega_{2\varepsilon_0} \setminus \partial\Omega$ such that

$$\inf_{\Omega \setminus \Omega_{2\varepsilon_0}} \bar{G}(z_1^\gamma, \cdot) = \inf_{(\Omega \setminus B_\rho(z_1^\gamma)) \setminus \Omega_{2\varepsilon_0}} \bar{G}(z_1^\gamma, \cdot) = \inf_{\partial\Omega_{2\varepsilon_0} \setminus \partial\Omega} \bar{G}(z_1^\gamma, \cdot) = \bar{G}(z_1^\gamma, \tilde{z}_2^\gamma).$$

The strong maximum principle also implies that $\partial_\nu \bar{G}(z_1^0, \cdot) \leq -\alpha_0$ on $\partial\Omega$, for some α_0 . Consequently, there exists $\beta(\varepsilon_0)$ such that

$$\bar{G}(z_1^\gamma, \tilde{z}_2^\gamma) - \bar{G}(z_1^\gamma, \tilde{\pi}_{\varepsilon_0} \tilde{z}_2^\gamma) \geq \beta(\varepsilon_0).$$

We conclude that

$$\begin{aligned} \mathcal{H}_\gamma(z_1^\gamma, z_2^\gamma) &= \min_{\mathcal{M}} \mathcal{H}_\gamma = h(z_1^\gamma) + 2\gamma \bar{G}(z_1^\gamma, z_2^\gamma) + \gamma^2 h(z_2^\gamma) \geq h(z_1^\gamma) + 2\gamma \bar{G}(z_1^\gamma, \tilde{z}_2^\gamma) + \gamma^2 h(z_2^\gamma) \\ &= \mathcal{H}_\gamma(z_1^\gamma, \pi_{\varepsilon_0} \tilde{z}_2^\gamma) + \gamma \{2[\bar{G}(z_1^\gamma, \tilde{z}_2^\gamma) - \bar{G}(z_1^\gamma, \pi_{\varepsilon_0} \tilde{z}_2^\gamma)] + \gamma[h(z_2^\gamma) - h(\pi_{\varepsilon_0} \tilde{z}_2^\gamma)]\}. \end{aligned}$$

Let $0 < \gamma''_\eta \leq \gamma'_\eta$ be such that

$$\gamma|h(z_2^\gamma) - h(\pi_{\varepsilon_0} \tilde{z}_2^\gamma)| < \beta(\varepsilon_0)$$

for all $0 < \gamma < \gamma''_\eta$. Then, for all $0 < \gamma < \gamma''_\eta$ we obtain that

$$\mathcal{H}_\gamma(z_1^\gamma, z_2^\gamma) = \min_{\mathcal{M}} \mathcal{H}_\gamma > \mathcal{H}_\gamma(z_1^\gamma, \pi_{\varepsilon_0} \tilde{z}_2^\gamma) + \gamma\beta(\varepsilon_0) > \mathcal{H}_\gamma(z_1^\gamma, z_2^\gamma),$$

a contradiction.

Claim 4. $h(z_1^0) = \min_\Omega h$.

Arguing by contradiction, assume that $h(z_1^\gamma) - \min_\Omega h \geq \eta_0 > 0$ for all sufficiently small values of γ . We write

$$\begin{aligned} \mathcal{H}_\gamma(z_1^\gamma, z_2^\gamma) &= h(z_1^\gamma) + 2\gamma \bar{G}(z_1^\gamma, z_2^\gamma) + \gamma^2 h(z_2^\gamma) \\ &= h(\underline{z}) + 2\gamma \bar{G}(\underline{z}, z_2^\gamma) + \gamma^2 h(z_2^\gamma) + h(z_1^\gamma) - h(\underline{z}) + 2\gamma[\bar{G}(z_1^\gamma, z_2^\gamma) - \bar{G}(\underline{z}, z_2^\gamma)] \\ &\geq \mathcal{H}_\gamma(\underline{z}, z_2^\gamma) + \eta_0 + 2\gamma[\bar{G}(z_1^\gamma, z_2^\gamma) - \bar{G}(\underline{z}, z_2^\gamma)], \end{aligned}$$

where $\underline{z} \in \Omega$ is a harmonic center for Ω . For γ sufficiently small we have $2\gamma|\bar{G}(z_1^\gamma, z_2^\gamma) - \bar{G}(\underline{z}, z_2^\gamma)| \leq \eta_0/2$. It follows that

$$\mathcal{H}_\gamma(z_1^\gamma, z_2^\gamma) \geq \mathcal{H}_\gamma(\underline{z}, z_2^\gamma) + \frac{\eta_0}{2} \geq \min_{\mathcal{M}} \mathcal{H}_\gamma + \frac{\eta_0}{2},$$

a contradiction. We conclude that z_1^0 is a harmonic center for Ω , and Part (i) is established.

Claim 6. We denote by $\pi_{\varepsilon_0} : \Omega_{\varepsilon_0} \rightarrow \partial\Omega$ the standard projection. We recall from [3], p. 204 that the expansion

$$h(z - d_z \nu(z)) = \frac{1}{2\pi} \log(2d_z) + o(d_z) \quad (7.2)$$

holds true for every $z \in \Omega_{\varepsilon_0}$. We also note that in view of the mean value theorem we may write

$$\bar{G}(z_1^\gamma, z) = \bar{G}(z_1^\gamma, \pi_{\varepsilon_0} z - d_z \nu) = -\partial \bar{G}_\nu(z_1^\gamma, \pi_{\varepsilon_0} z) d_z + o(d_z). \quad (7.3)$$

Let $p \in \partial\Omega$. Then, $\mathcal{H}_\gamma(z_1^\gamma, z_2^\gamma) \leq \mathcal{H}_\gamma(z_1^\gamma, p - d_{z_2^\gamma} \nu(p))$ and we deduce that

$$2\bar{G}(z_1^\gamma, z_2^\gamma) + \gamma h(z_2^\gamma) \leq 2\bar{G}(z_1^\gamma, p - d_{z_2^\gamma} \nu(p)) + \gamma h(p - d_{z_2^\gamma} \nu(p))$$

In view of (7.2) and using $z_2^\gamma = \pi_{\varepsilon_0} z_2^\gamma - d_{z_2^\gamma} \nu(z_2^\gamma)$, we derive

$$\bar{G}(z_1^\gamma, z_2^\gamma) + \frac{\gamma}{2} \left[\frac{1}{2\pi} \log(2d_{z_2^\gamma}) + o(d_{z_2^\gamma}) \right] \leq \bar{G}(z_1^\gamma, p - d_{z_2^\gamma} \nu(p)) + \frac{\gamma}{2} \left[\frac{1}{2\pi} \log(2d_{z_2^\gamma}) + o(d_{z_2^\gamma}) \right].$$

from which we derive that

$$\bar{G}(z_1^\gamma, z_2^\gamma) \leq \bar{G}(z_1^\gamma, p - d_{z_2^\gamma} \nu(p)) + o(d_{z_2^\gamma}).$$

Finally, in view of (7.3) we deduce

$$-\partial \bar{G}_\nu(z_1^\gamma, \pi_{\varepsilon_0} z_2^\gamma) d_{z_2^\gamma} \leq \partial_\nu \bar{G}(z_1^\gamma, p) d_{z_2^\gamma} + o(d_{z_2^\gamma})$$

and finally

$$\partial \bar{G}_\nu(z_1^\gamma, \pi_{\varepsilon_0} z_2^\gamma) \geq \partial_\nu \bar{G}(z_1^\gamma, p).$$

Letting $\gamma \rightarrow 0^+$ we conclude that

$$\partial \bar{G}_\nu(z_1^0, z_2^0) \geq \partial_\nu \bar{G}(z_1^0, p)$$

for any $p \in \partial\Omega$. Now Part (ii) is completely established. \square

Now we can conclude the proof of Theorem 2.2.

Proof of Theorem 2.2-(ii). We observe that the reduced functional $K_\varepsilon(z_1, z_2)$ is of the form

$$K_\varepsilon(z_1, z_2) = I_\varepsilon(W_\varepsilon) = C_1(\varepsilon) a_1^2 \mathcal{H}_{a_2/a_1}(z_1, z_2) + C_2(\varepsilon)$$

and that

$$\frac{a_2}{a_1} = \frac{\gamma + O\left(\frac{1}{|\ln \varepsilon|}\right)}{1 + O\left(\frac{1}{|\ln \varepsilon|}\right)} = \gamma + O\left(\frac{1}{|\ln \varepsilon|}\right).$$

In view of Lemma 7.1 we conclude that for any $\eta > 0$ there exist a sufficiently small $\gamma_0 > 0$ with the property that for any $\gamma \in (0, \gamma_0)$ there exists $\varepsilon_\gamma > 0$ such that the peak points (z_1, z_2) of the constructed solution u_ε satisfy $d(z_1, \underline{z}) < \eta$ and $d(z_2, \partial\Omega) < \eta$ for all $\varepsilon \in (0, \varepsilon_\gamma)$. \square

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